# Practice Problem Set #2 Solutions

1. Determine the unilateral Laplace transform of the following signals:

(a) $x(t) = u(t-2)$
(b) $x(t) = u(t+2)$
(c) $x(t) = e^{-2t}u(t+1)$
(d) $x(t) = e^{2t}u(-t+2)$
(e) $x(t) = \sin(\omega_o t)$
(f) $x(t) = u(t) - u(t-2)$
(g) $x(t) = \begin{cases} \sin(\pi t), & 0 < t < 1 \\ 0, & \text{otherwise} \end{cases}$

(a) $x(t) = u(t-2)$

$$\begin{aligned} X(s) &= \int_{0^-}^{\infty} x(t)e^{-st}\,dt \\ &= \int_{0^-}^{\infty} u(t-2)e^{-st}\,dt \\ &= \int_{2}^{\infty} e^{-st}\,dt \\ &= \frac{e^{-2s}}{s} \end{aligned}$$

(b) $x(t) = u(t+2)$

$$\begin{aligned} X(s) &= \int_{0^-}^{\infty} u(t+2)e^{-st}\,dt \\ &= \int_{0^-}^{\infty} e^{-st}\,dt \\ &= \frac{1}{s} \end{aligned}$$

(c) $x(t) = e^{-2t}u(t+1)$

$$
\begin{aligned}
X(s) &= \int_{0^-}^{\infty} e^{-2t}u(t+1)e^{-st}\,dt \\
&= \int_{0^-}^{\infty} e^{-t(s+2)}\,dt \\
&= \frac{1}{s+2}
\end{aligned}
$$

(d) $x(t) = e^{2t}u(-t+2)$

$$
\begin{aligned}
X(s) &= \int_{0^-}^{\infty} e^{2t}u(-t+2)e^{-st}\,dt \\
&= \int_{0^-}^{2} e^{t(2-s)}\,dt \\
&= \frac{e^{2(2-s)}-1}{2-s}
\end{aligned}
$$

(e) $x(t) = \sin(\omega_o t)$

$$
\begin{aligned}
X(s) &= \int_{0^-}^{\infty} \frac{1}{2j}\left(e^{j\omega_o t} - e^{-j\omega_o t}\right)e^{-st}\,dt \\
&= \frac{1}{2j}\left[\int_{0^-}^{\infty} e^{t(j\omega_o - s)}\,dt - \int_{0^-}^{\infty} e^{-t(j\omega_o + s)}\,dt\right] \\
&= \frac{1}{2j}\left[\frac{-1}{j\omega_o - s} - \frac{1}{j\omega_o + s}\right] \\
&= \frac{\omega_o}{s^2+\omega_o^2}
\end{aligned}
$$

(f) $x(t) = u(t) - u(t-2)$

$$
\begin{aligned}
X(s) &= \int_{0^-}^{2} e^{-st}\,dt \\
&= \frac{1-e^{-2s}}{s}
\end{aligned}
$$

(g) $x(t) = \begin{cases} \sin(\pi t), & 0 < t < 1 \\ 0, & \text{otherwise} \end{cases}$

$$X(s) = \int_{0^-}^{1} \frac{1}{2j}\left(e^{j\pi t} - e^{-j\pi t}\right) e^{-st}\, dt = \frac{\pi(1+e^{-s})}{s^2+\pi^2}$$

2. Use the Laplace transform tables and properties to obtain the Laplace transform of the following:

(a) $x(t) = \frac{d}{dt}\{te^{-t}u(t)\}$
(b) $x(t) = tu(t) * \cos(2\pi t)u(t)$
(c) $x(t) = t^3 u(t)$
(d) $x(t) = u(t-1) * e^{-2t}u(t-1)$
(e) $x(t) = \int_0^t e^{-3\tau}\cos(2\tau)d\tau$
(f) $x(t) = t\frac{d}{dt}(e^{-t}\cos(t)u(t))$

(a) $x(t) = \frac{d}{dt}\{te^{-t}u(t)\}$

$$a(t) = te^{-t}u(t) \xleftrightarrow{\mathcal{L}_u} A(s) = \frac{1}{(s+1)^2}$$

$$x(t) = \frac{d}{dt}a(t) \xleftrightarrow{\mathcal{L}_u} X(s) = \frac{s}{(s+1)^2}$$

(b) $x(t) = tu(t) * \cos(2\pi t)u(t)$

$$a(t) = tu(t) \xleftrightarrow{\mathcal{L}_u} A(s) = \frac{1}{s^2}$$

$$b(t) = \cos(2\pi t)u(t) \xleftrightarrow{\mathcal{L}_u} \frac{s}{s^2+4\pi^2}$$

$$x(t) = a(t) * b(t) \xleftrightarrow{\mathcal{L}_u} X(s) = A(s)B(s)$$

$$X(s) = \frac{1}{s^2(s^2+4\pi^2)}$$

(c) $x(t) = t^3 u(t)$

$$\begin{aligned}
a(t) = tu(t) \quad &\overset{\mathcal{L}_u}{\longleftrightarrow} \quad A(s) = \frac{1}{s^2} \\
b(t) = -ta(t) \quad &\overset{\mathcal{L}_u}{\longleftrightarrow} \quad B(s) = \frac{d}{ds}A(s) = \frac{-2}{s^3} \\
x(t) = -tb(t) \quad &\overset{\mathcal{L}_u}{\longleftrightarrow} \quad X(s) = \frac{d}{ds}B(s) = \frac{6}{s^4}
\end{aligned}$$

(d) $x(t) = u(t-1) * e^{-2t}u(t-1)$

$$\begin{aligned}
a(t) = u(t) \quad &\overset{\mathcal{L}_u}{\longleftrightarrow} \quad A(s) = \frac{1}{s} \\
b(t) = a(t-1) \quad &\overset{\mathcal{L}_u}{\longleftrightarrow} \quad B(s) = \frac{e^{-s}}{s} \\
c(t) = e^{-2t}u(t) \quad &\overset{\mathcal{L}_u}{\longleftrightarrow} \quad C(s) = \frac{1}{s+2} \\
d(t) = e^{-2}c(t-1) \quad &\overset{\mathcal{L}_u}{\longleftrightarrow} \quad D(s) = \frac{e^{-(s+2)}}{s+2} \\
x(t) = b(t) * d(t) \quad &\overset{\mathcal{L}_u}{\longleftrightarrow} \quad X(s) = B(s)D(s) \\
X(s) \quad &= \quad \frac{e^{-2(s+1)}}{s(s+2)}
\end{aligned}$$

(e) $x(t) = \int_0^t e^{-3\tau}\cos(2\tau)d\tau$

$$\begin{aligned} a(t) = e^{-3t}\cos(2t)u(t) &\overset{\mathcal{L}_u}{\longleftrightarrow} A(s) = \frac{s+3}{(s+3)^2+4} \\ \int_{-\infty}^{t} a(\tau)d\tau &\overset{\mathcal{L}_u}{\longleftrightarrow} \frac{1}{s}\int_{-\infty}^{0^-} a(\tau)d\tau + \frac{A(s)}{s} \\ X(s) &= \frac{s+3}{s((s+3)^2+4)} \end{aligned}$$

(f) $x(t) = t\frac{d}{dt}\left(e^{-t}\cos(t)u(t)\right)$

$$\begin{aligned} a(t) = e^{-t}\cos(t)u(t) &\overset{\mathcal{L}_u}{\longleftrightarrow} A(s) = \frac{s+1}{(s+1)^2+1} \\ b(t) = \frac{d}{dt}a(t) &\overset{\mathcal{L}_u}{\longleftrightarrow} B(s) = \frac{s(s+1)}{(s+1)^2+1} \\ x(t) = tb(t) &\overset{\mathcal{L}_u}{\longleftrightarrow} X(s) = -\frac{d}{ds}B(s) \\ X(s) &= \frac{-s^2-4s-2}{(s^2+2s+2)^2} \end{aligned}$$

3. Use the tables of transforms and properties to determine the time signals that correspond to the following bilateral Laplace transforms:

**(a)** $X(s) = e^{5s}\frac{1}{s+2}$ **with ROC** $\mathrm{Re}(s) < -2$

**(b)** $X(s) = \frac{d^2}{ds^2}\left(\frac{1}{s-3}\right)$ **with ROC** $\mathrm{Re}(s) > 3$

**(c)** $X(s) = s\left(\frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-2s}}{s}\right)$ **with ROC** $\mathrm{Re}(s) < 0$

**(d)** $X(s) = s^{-2}\frac{d}{ds}\left(\frac{e^{-3s}}{s}\right)$ **with ROC** $\mathrm{Re}(s) > 0$

(a) $X(s) = e^{5s}\frac{1}{s+2}$ with ROC $\text{Re}(s) < -2$

$$
\begin{aligned}
&\text{(left-sided)} \\
A(s) = \frac{1}{s+2} \quad &\overset{\mathcal{L}}{\longleftrightarrow} \quad a(t) = -e^{-2t}u(-t) \\
X(s) = e^{5s}A(s) \quad &\overset{\mathcal{L}}{\longleftrightarrow} \quad x(t) = a(t+5) = -e^{-2(t+5)}u(-(t+5))
\end{aligned}
$$

(b) $X(s) = \frac{d^2}{ds^2}\left(\frac{1}{s-3}\right)$ with ROC $\text{Re}(s) > 3$

$$
\begin{aligned}
&\text{(right-sided)} \\
A(s) = \frac{1}{s-3} \quad &\overset{\mathcal{L}}{\longleftrightarrow} \quad a(t) = e^{3t}u(t) \\
X(s) = \frac{d^2}{ds^2}A(s) \quad &\overset{\mathcal{L}}{\longleftrightarrow} \quad x(t) = t^2e^{3t}u(t)
\end{aligned}
$$

(c) $X(s) = s\left(\frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-2s}}{s}\right)$ with ROC $\text{Re}(s) < 0$

$$
\begin{aligned}
&\text{(left-sided)} \\
x(t) &= \frac{d}{dt}\left(-tu(-t) + tu(-t-1) + u(-t-2)\right) \\
x(t) &= -u(-t) + u(-t-1) - \delta(t+2)
\end{aligned}
$$

(d) $X(s) = s^{-2}\frac{d}{ds}\left(\frac{e^{-3s}}{s}\right)$ with ROC $\text{Re}(s) > 0$

$$
\begin{aligned}
&\text{(right-sided)} \\
A(s) = \frac{1}{s} \quad &\overset{\mathcal{L}}{\longleftrightarrow} \quad a(t) = u(t) \\
B(s) = e^{-3s}A(s) \quad &\overset{\mathcal{L}}{\longleftrightarrow} \quad b(t) = a(t-3) = u(t-3) \\
C(s) = \frac{d}{ds}B(s) \quad &\overset{\mathcal{L}}{\longleftrightarrow} \quad c(t) = -tb(t) = -tu(t-3) \\
D(s) = \frac{1}{s} \quad &\overset{\mathcal{L}}{\longleftrightarrow} \quad d(t) = \int_{-\infty}^{t} c(\tau)d\tau \\
&\overset{\mathcal{L}}{\longleftrightarrow} \quad d(t) = -\int_{3}^{t} \tau d\tau = -\frac{1}{2}(t^2 - 9) \\
X(s) = \frac{1}{s}D(s) \quad &\overset{\mathcal{L}}{\longleftrightarrow} \quad x(t) = \int_{-\infty}^{t} d(\tau)d\tau \\
&\overset{\mathcal{L}}{\longleftrightarrow} \quad x(t) = -\frac{1}{2}\int_{3}^{t}(\tau^2 - 9)d\tau \\
&\overset{\mathcal{L}}{\longleftrightarrow} \quad x(t) = \left[-\frac{1}{6}(t^3 - 27) + \frac{9}{2}(t-3)\right]u(t-3)
\end{aligned}
$$

4. Evaluate the frequency-domain representations of the following signals:

(a) $x(t) = e^{-2t}u(t-3)$
(b) $x(t) = e^{-4|t|}$
(c) $x(t) = te^{-t}u(t)$
(d) $x(t) = \sum_{m=0}^{\infty} a^m \delta(t-m)$, $|a| < 1$

(a) $x(t) = e^{-2t}u(t-3)$

$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt \\ &= \int_{3}^{\infty} e^{-2t}e^{-j\omega t}\,dt \\ &= \frac{e^{-3(2+j\omega)}}{2+j\omega} \end{aligned}$$

(b) $x(t) = e^{-4|t|}$

$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{\infty} e^{-4|t|}e^{-j\omega t}\,dt \\ &= \int_{0}^{\infty} e^{-4t}e^{-j\omega t}\,dt + \int_{-\infty}^{0} e^{4t}e^{-j\omega t}\,dt \\ &= \frac{8}{16+\omega^2} \end{aligned}$$

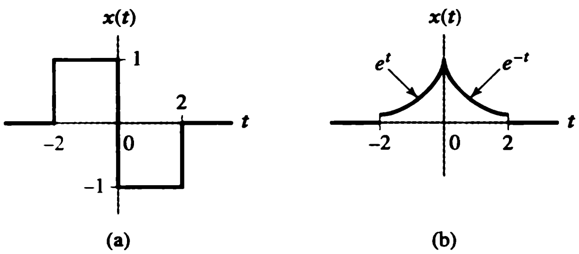
(c) $x(t) = te^{-t}u(t)$

$$\begin{aligned} X(j\omega) &= \int_{0}^{\infty} te^{-t}e^{-j\omega t}\,dt \\ &= \frac{1}{(1+j\omega)^2} \end{aligned}$$

(d) $x(t) = \sum_{m=0}^{\infty} a^m \delta(t-m), \quad |a| < 1$

$$\begin{aligned} X(j\omega) &= \int_{0}^{\infty} (\sum_{m=0}^{\infty} a^m \delta(t-m))e^{-j\omega t}\,dt \\ &= \sum_{m=0}^{\infty} (ae^{-j\omega})^m \\ &= \frac{1}{1-ae^{-j\omega}} \end{aligned}$$

5. Evaluate the frequency-domain representations of the shown signals:




(a) x(t)= u(t+2)-2u(t)+u(t-2) and evaluate Fourier transforms from table. Alternatively:

$$X(j\omega) = \int_{-1}^{0} e^{-j\omega t}\,dt - \int_{0}^{1} e^{-j\omega t}\,dt$$
$$= \frac{2\cos(\omega) - 2}{j\omega}$$

$$X(j\omega) = \begin{cases} \frac{2\cos(\omega)-2}{j\omega} & \omega \neq 0 \\ 0 & \omega = 0 \end{cases}$$

(b) x(t)= exp(-|t|) (u(t+2)-u(t-2)) and evaluate from tables and use convolution property. Alternatively:

$$X(j\omega) = \int_{-2}^{0} e^{t} e^{-j\omega t}\,dt + \int_{0}^{2} e^{-t} e^{-j\omega t}\,dt$$
$$= \frac{1 - e^{-(1-j\omega)2}}{1 - j\omega} + \frac{1 - e^{-(1+j\omega)2}}{1 + j\omega}$$
$$= \frac{2 - 2e^{-2}\cos(2\omega) + 2\omega e^{-2}\sin(2\omega)}{1 + \omega^2}$$

6. Use the Fourier transform tables and properties to obtain the Fourier transform of the following signals:

(a) $x(t) = \sin(2\pi t)e^{-t}u(t)$

(b) $x(t) = te^{-3|t-1|}$

(c) $x(t) = \left[\frac{2\sin(3\pi t)}{\pi t}\right]\left[\frac{\sin(2\pi t)}{\pi t}\right]$

(d) $x(t) = \frac{d}{dt}(te^{-2t}\sin(t)u(t))$

(e) $x(t) = \int_{-\infty}^{t} \frac{\sin(2\pi\tau)}{\pi\tau}d\tau$

(f) $x(t) = e^{-t+2}u(t-2)$

(g) $x(t) = \left(\frac{\sin(t)}{\pi t}\right) * \frac{d}{dt}\left[\left(\frac{\sin(2t)}{\pi t}\right)\right]$

(a) $x(t) = \sin(2\pi t)e^{-t}u(t)$

$$\begin{aligned} x(t) &= \sin(2\pi t)e^{-t}u(t) \\ &= \frac{1}{2j}e^{j2\pi t}e^{-t}u(t) - \frac{1}{2j}e^{-j2\pi t}e^{-t}u(t) \end{aligned}$$

$$\begin{aligned} e^{-t}u(t) &\overset{FT}{\longleftrightarrow} \frac{1}{1+j\omega} \\ e^{j2\pi t}s(t) &\overset{FT}{\longleftrightarrow} S(j(\omega-2\pi)) \\ X(j\omega) &= \frac{1}{2j}\left[\frac{1}{1+j(\omega-2\pi)} - \frac{1}{1+j(\omega+2\pi)}\right] \end{aligned}$$

(b) $x(t) = te^{-3|t-1|}$

$$\begin{aligned} e^{-3|t|} &\overset{FT}{\longleftrightarrow} \frac{6}{9+\omega^2} \\ s(t-1) &\overset{FT}{\longleftrightarrow} e^{-j\omega}S(j\omega) \\ tw(t) &\overset{FT}{\longleftrightarrow} j\frac{d}{d\omega}W(j\omega) \\ X(j\omega) &= j\frac{d}{d\omega}\left[e^{-j\omega}\frac{6}{9+\omega^2}\right] \\ &= \frac{6e^{-j\omega}}{9+\omega^2} - \frac{12j\omega^{-j\omega}}{(9+\omega^2)^2} \end{aligned}$$

(c) $x(t) = \left[\frac{2\sin(3\pi t)}{\pi t}\right]\left[\frac{\sin(2\pi t)}{\pi t}\right]$

$$\begin{aligned} \frac{\sin(Wt)}{\pi t} &\overset{FT}{\longleftrightarrow} \begin{cases} 1 & \omega \le W \\ 0, & \text{otherwise} \end{cases} \\ s_1(t)s_2(t) &\overset{FT}{\longleftrightarrow} \frac{1}{2\pi}S_1(j\omega) * S_2(j\omega) \end{aligned}$$

$$X(j\omega) = \begin{cases} 5 - \frac{|\omega|}{\pi} & \pi < |\omega| \le 5\pi \\ 4 & |\omega| \le \pi \\ 0 & \text{otherwise} \end{cases}$$

(d)

$$\begin{aligned} x(t) &= \frac{d}{dt}te^{-2t}\sin(t)u(t) \\ &= \frac{d}{dt}te^{-2t}u(t)\frac{e^{jt}-e^{-jt}}{2j} \end{aligned}$$

$$te^{-2t}u(t) \overset{FT}{\longleftrightarrow} \frac{1}{(2+j\omega)^2}$$

7. Replace the time variable "t" with the frequency variable "$\Omega$" in all signals in problems 4, 5 and 6 and repeat to obtain the inverse Fourier transform of these signals.

Solution: Use the duality property to do that in one step.