Ultrasound Bioinstrumentation

Topic 1 (lecture 2)

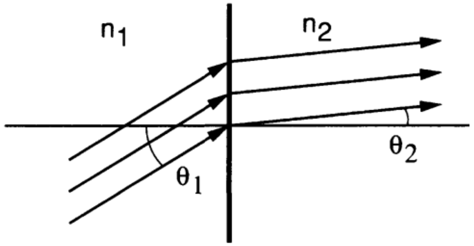
Fundamentals of Scalar Diffraction Theory

CAIRO UNIVERSITY

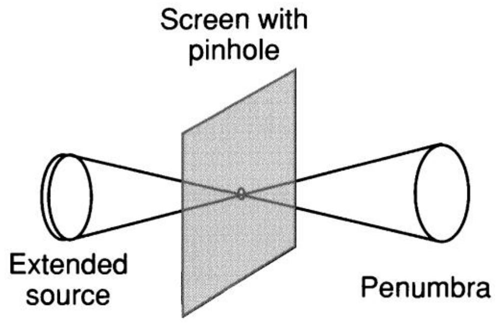
Fundamentals of Scalar Diffraction Theory

- Basic concepts
  - History
  - Scalar wave theory
  - Angular spectrum
  - Propagation as a linear spatial filter
  - Fresnel approximation
  - Fraunhofer approximation
  - Applications

Basics

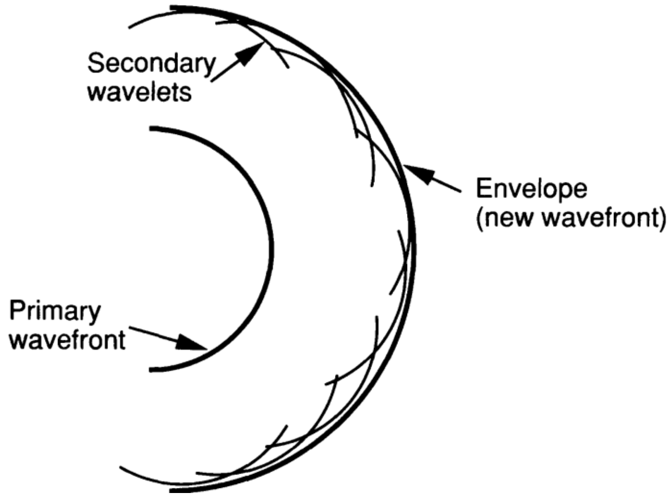


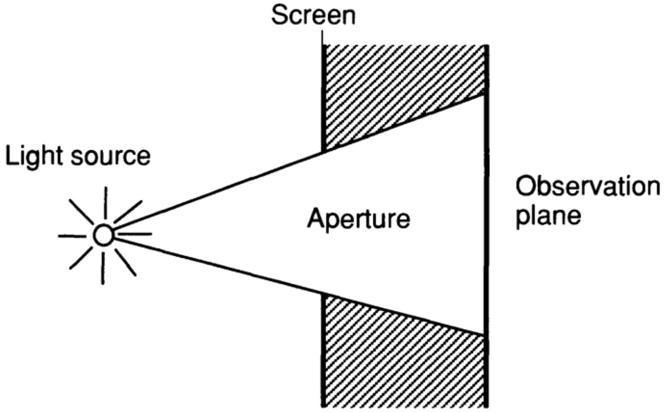
- Refraction (Snell's law)
- Diffraction definition (Sommerfeld):
  - Any deviation of light rays from rectilinear paths which cannot be interpreted as reflection or refraction
- Not penumbra effect
  - No bending of rays



Huygens Theory of Light

- Point sources





Major Observations

- Interfererence (Young)
  - Light can add to darkness
- Wavelet interference (Fresnel)
  - Bright spot at the center of the shadow of an opaque disk (Poisson’s spot)
- Maxwell equations
- Rayleigh-Sommerfeld diffraction theory

Maxwell Equations

$$\nabla \times \vec{\mathcal{E}} = -\mu \frac{\partial \vec{\mathcal{H}}}{\partial t}$$

$$\nabla \times \vec{\mathcal{H}} = \epsilon \frac{\partial \vec{\mathcal{E}}}{\partial t}$$

$$\nabla \cdot \epsilon \vec{\mathcal{E}} = 0$$

$$\nabla \cdot \mu \vec{\mathcal{H}} = 0.$$

Assumptions about Medium

- Linear
- Isotropic
  - independent of direction of polarization
- Homogeneous
  - Constant permittivity
- Nondispersive
  - Permittivity is independent of wavelength

Scalar Wave Equation

$$\nabla^2 \vec{\mathcal{E}} - \frac{n^2}{c^2}\frac{\partial^2 \vec{\mathcal{E}}}{\partial t^2} = 0$$

$$\nabla^2 \vec{\mathcal{H}} - \frac{n^2}{c^2}\frac{\partial^2 \vec{\mathcal{H}}}{\partial t^2} = 0.$$

$$\nabla^2 u(P,t) - \frac{n^2}{c^2}\frac{\partial^2 u(P,t)}{\partial t^2} = 0,$$

Validity of Scalar Theory

- Aperture is large compared to wavelength
  - Comparison to lumped circuit components
- Observations are sufficiently far away from the aperture (many wavelengths)

Helmholtz Equation

- Plug monochromatic wave into scalar wave equation:

$$u(P, t) = Re\{U(P)\exp(-j2\pi\nu t)\},$$

$$(\nabla^2 + k^2)U = 0.$$

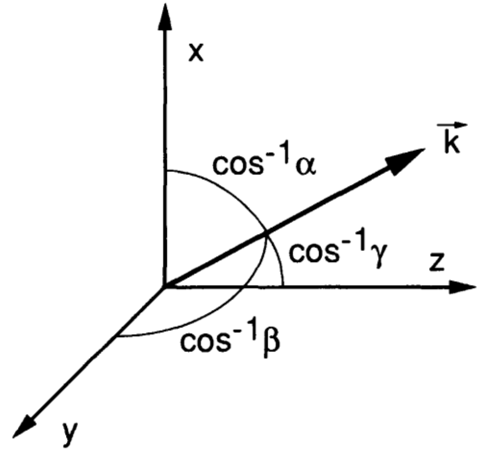
- Here wave number $k = 2\pi n \frac{\nu}{c} = \frac{2\pi}{\lambda}$

Plane Waves

- Eigenfunctions of propagation

$$p(x, y, z; t) = \exp[j(\vec{k} \cdot \vec{r} - 2\pi \nu t)]$$

$$\vec{k} = \frac{2\pi}{\lambda}(\alpha \hat{x} + \beta \hat{y} + \gamma \hat{z})$$



- Plane wave at z=0

$$\exp[j2\pi(f_X x + f_Y y)]$$

$$\alpha = \lambda f_X \quad \beta = \lambda f_Y \quad \gamma = \sqrt{1 - (\lambda f_X)^2 - (\lambda f_Y)^2}.$$

Angular Spectrum

- 2D Fourier transform of aperture

$$A(f_X, f_Y; 0) = \iint_{-\infty}^{\infty} U(x, y, 0) \exp[-j2\pi(f_X x + f_Y y)]\, dx\, dy.$$

$$U(x, y, 0) = \iint_{-\infty}^{\infty} A(f_X, f_Y; 0) \exp[j2\pi(f_X x + f_Y y)]\, df_X\, df_Y.$$

- Angular spectrum

$$A\left(\frac{\alpha}{\lambda}, \frac{\beta}{\lambda}; 0\right) = \iint_{-\infty}^{\infty} U(x, y, 0) \exp\left[-j2\pi\left(\frac{\alpha}{\lambda}x + \frac{\beta}{\lambda}y\right)\right] dx\, dy$$

Propagation of Angular Spectrum

$$A\left(\frac{\alpha}{\lambda}, \frac{\beta}{\lambda}; z\right) = A\left(\frac{\alpha}{\lambda}, \frac{\beta}{\lambda}; 0\right)\exp(-\mu z)$$

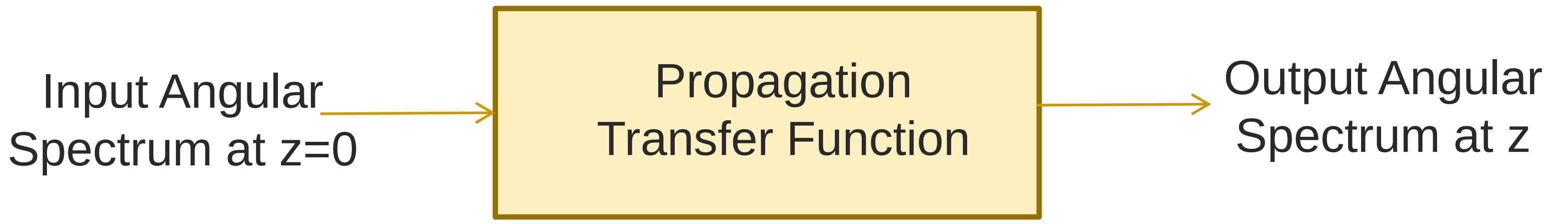
$$\mu = \frac{2\pi}{\lambda}\sqrt{\alpha^2 + \beta^2 - 1}.$$

Propagation as a Linear Spatial Filter

- Free space propagation transfer function

$$H(f_X, f_Y) = \begin{cases} \exp\left[j2\pi \frac{z}{\lambda} \sqrt{1 - (\lambda f_X)^2 - (\lambda f_Y)^2} \right] & \sqrt{f_X^2 + f_Y^2} < \frac{1}{\lambda} \\ 0 & \text{otherwise.} \end{cases}$$

$$\alpha = \lambda f_X \quad \beta = \lambda f_Y$$



Fresnel Approximation

- Paraxial (near field) approximation

$$\sqrt{1-(\lambda f_X)^2-(\lambda f_Y)^2} \approx 1-\frac{(\lambda f_X)^2}{2}-\frac{(\lambda f_Y)^2}{2},$$

$$H(f_X, f_Y) = e^{jkz}\exp\left[-j\pi\lambda z\left(f_X^2+f_Y^2\right)\right].$$

$$h(x, y) = \frac{e^{jkz}}{j\lambda z}\exp\left[\frac{jk}{2z}\left(x^2+y^2\right)\right].$$

Fraunhofer Approximation

- Far field approximation

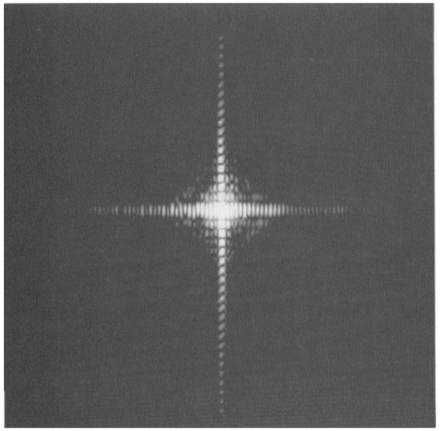
$$z \gg \frac{k(\xi^2 + \eta^2)_{\max}}{2}$$

$$U(x, y) = \frac{e^{jkz} e^{j\frac{k}{2z}(x^2+y^2)}}{j\lambda z} \iint_{-\infty}^{\infty} U(\xi, \eta) \exp\left[-j\frac{2\pi}{\lambda z}(x\xi + y\eta)\right] d\xi \, d\eta.$$

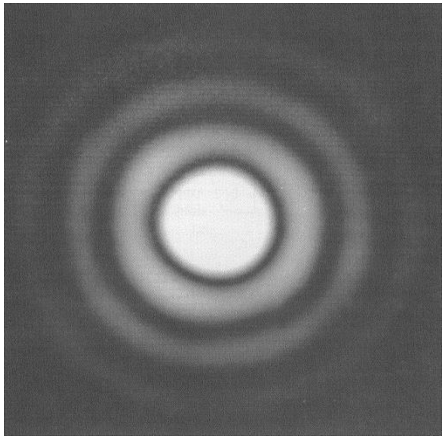
- Field = $\Im$\{ Aperture \}

Examples

- Rectangular aperture



- Circular aperture



Problem Assignments

- Problems: 3.5, 4.7, 4.9, 4.10