54, and 64 with $\beta = 5.48$. First note that in all cases, the 1024-point DFT gives a smooth result when the points are connected by straight lines. In Figure 10.10(a), where $L = 32$, the two sinusoidal components are not resolved, and, of course, increasing the DFT length will only result in a smoother curve. As the window length increases, however, we see steady improvement in our ability to distinguish the two frequencies and the approximate amplitudes of each sinusoidal component. Note that the 1024-point DFT in Figure 10.10(d) would be much more effective for precisely locating the peak of the windowed Fourier transform than the coarsely sampled DFT in Figure 10.8(b), which is also computed with a 64-point Kaiser window. Note also that the amplitudes of the two peaks in Figure 10.10 are very close to being in the correct ratio of 0.75 to 1.

10.3 THE TIME-DEPENDENT FOURIER TRANSFORM

The previous section illustrated the use of the DFT for obtaining a frequency-domain representation of a signal composed of sinusoidal components. In that discussion, we assumed that the frequencies of the cosines did not change with time so that no matter how long the window, the signal properties would be the same from the beginning to the end of the window. Often, in practical applications of sinusoidal signal models, the signal properties (amplitudes, frequencies, and phases) will change with time. For example, nonstationary signal models of this type are required to describe radar, sonar, speech, and data communication signals. A single DFT estimate is not sufficient to describe such signals, and as a result, we are led to the concept of the *time-dependent Fourier transform,* also referred to as the short-time Fourier transform.[1]

The time-dependent Fourier transform of a signal $x[n]$ is defined as

$$X[n, \lambda) = \sum_{m=-\infty}^{\infty} x[n+m]w[m]e^{-j\lambda m}, \tag{10.18}$$

where $w[n]$ is a window sequence. In the time-dependent Fourier representation, the one-dimensional sequence $x[n]$, a function of a single discrete variable, is converted into a two-dimensional function of the time variable n, which is discrete, and the frequency variable λ, which is continuous.[2] Note that the time-dependent Fourier transform is periodic in λ with period 2π, and therefore, we need consider only values of λ for $0 \le \lambda < 2\pi$ or any other interval of length 2π.

Equation (10.18) can be interpreted as the Fourier transform of the shifted signal $x[n+m]$, as viewed through the window $w[m]$. The window has a stationary origin, and as n changes, the signal slides past the window so that, at each value of n, a different portion of the signal is viewed.

[1]Further discussion of the time-dependent Fourier transform can be found in a variety of references, including Allen and Rabiner (1977), Rabiner and Schafer (1978), Crochiere and Rabiner (1983), and Nawab and Quatieri (1988).

[2]We denote the frequency variable of the time-dependent Fourier transform by λ to maintain a distinction from the frequency variable of the conventional discrete-time Fourier transform, which will be denoted ω. We use the mixed bracket–parenthesis notation $X[n, \lambda)$ as a reminder that n is a discrete variable and λ a continuous variable.

Example 10.9 Time-Dependent Fourier Transform of a Linear Chirp Signal

The relationship of the window to the shifted signal is illustrated in Figure 10.11 for the signal

$$x[n] = \cos(\omega_0 n^2), \qquad \omega_0 = 2\pi \times 7.5 \times 10^{-6}, \tag{10.19}$$

corresponding to a linear frequency modulation (i.e., the "instantaneous frequency" is $2\omega_0 n$). As we saw in Chapter 9 in the context of the chirp transform algorithm, a signal of this type is often referred to as a linear chirp. Typically, $w[m]$ in Eq. (10.18) has finite length around $m = 0$, so that $X[n, \lambda)$ displays the frequency characteristics of the signal around time n. For example, in Figure 10.12 we show a display of the magnitude of the

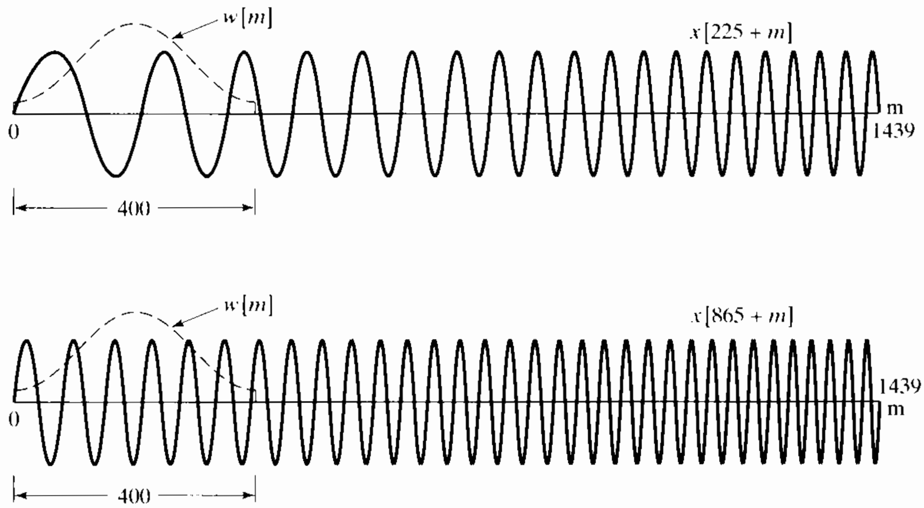

Figure 10.11 Two segments of the linear chirp signal $x[n] = \cos(\omega_0 n^2)$ with the window superimposed. $X[n, \lambda)$ at $n = 225$ is the discrete-time Fourier transform of the top trace multiplied by the window. $X[865, \lambda)$ is the discrete-time Fourier transform of the bottom trace multiplied by the window.

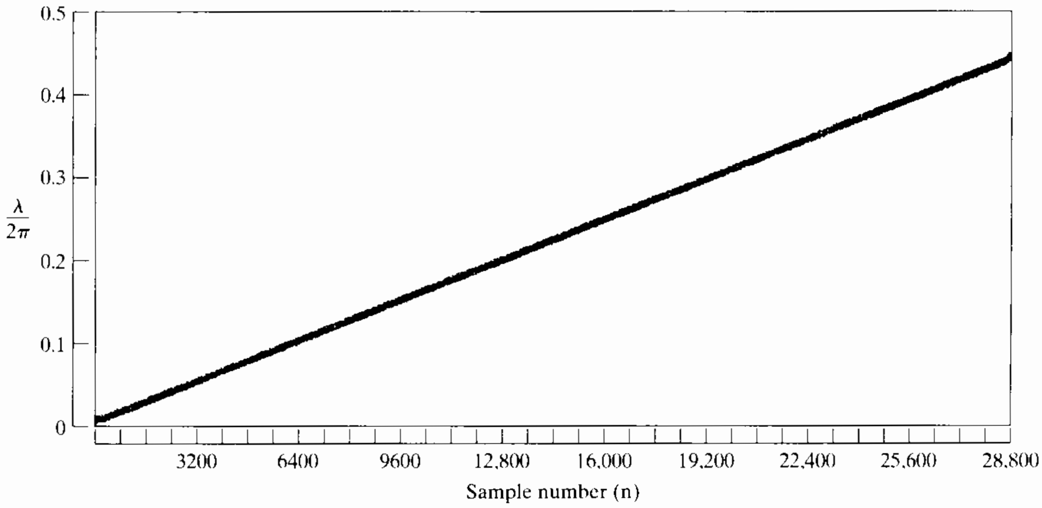

Figure 10.12 The magnitude of the time-dependent Fourier transform of $x[n] = \cos(\omega_0 n^2)$ using a Hamming window of length 400.

time-dependent Fourier transform of the signal of Eq. (10.19) and Figure 10.11 with $w[m]$ a Hamming window of length 400. In this display, referred to as a *spectrogram*, the vertical dimension $\lambda/2\pi$ is proportional to frequency and the horizontal dimension (n) is proportional to time. The magnitude of the time-dependent Fourier transform is represented by the darkness of the markings. In Figure 10.12, the linear progression of the frequency with time is clear.

Since $X[n, \lambda)$ is the discrete-time Fourier transform of $x[n+m]w[m]$, the time-dependent Fourier transform is invertible if the window has at least one nonzero sample. Specifically, from the Fourier transform synthesis equation (2.133),

$$x[n+m]w[m] = \frac{1}{2\pi}\int_0^{2\pi} X[n,\lambda)e^{j\lambda m}d\lambda, \qquad -\infty < m < \infty, \tag{10.20}$$

from which it follows that

$$x[n] = \frac{1}{2\pi w[0]}\int_0^{2\pi} X[n,\lambda)d\lambda \tag{10.21}$$

if $w[0] \neq 0$.[3] Not just the single sample $x[n]$, but all of the samples that are multiplied by nonzero samples of the window, can be recovered in a similar manner using Eq. (10.20).

A rearrangement of the sum in Eq. (10.18) leads to another useful interpretation of the time-dependent Fourier transform. If we make the substitution $m' = n + m$ in Eq. (10.18), then $X[n, \lambda)$ can be written as

$$X[n,\lambda) = \sum_{m'=-\infty}^{\infty} x[m']w[-(n-m')]e^{j\lambda(n-m')}. \tag{10.22}$$

Equation (10.22) can be interpreted as the convolution

$$X[n,\lambda) = x[n] * h_\lambda[n], \tag{10.23a}$$

where

$$h_\lambda[n] = w[-n]e^{j\lambda n}. \tag{10.23b}$$

From Eq. (10.23a), we see that the time-dependent Fourier transform as a function of n with λ fixed can be interpreted as the output of a linear time-invariant filter with impulse response $h_\lambda[n]$ or, equivalently, with frequency response

$$H_\lambda(e^{j\omega}) = W(e^{j(\lambda-\omega)}). \tag{10.24}$$

In general, a window that is nonzero for positive time will be called a *noncausal window*, since the computation of $X[n, \lambda)$ using Eq. (10.18) requires samples that *follow* sample n in the sequence. Equivalently, in the linear-filtering interpretation, the impulse response $h_\lambda[n] = w[-n]e^{j\lambda n}$ is noncausal.

In the definition of Eq. (10.18), the time origin of the window is held fixed and the signal is shifted past the interval of support of the window. This effectively redefines the time origin for Fourier analysis to be at sample n of the signal. Another possibility is to shift the window as n changes, keeping the time origin for Fourier analysis fixed at

[3]Since $X[n, \lambda)$ is periodic in λ with period 2π, the integration in Eqs. (10.20) and (10.21) can be over any interval of length 2π.

the original time origin of the signal. This leads to a definition for the time-dependent Fourier transform of the form

$$\check{X}[n, \lambda) = \sum_{m=-\infty}^{\infty} x[m]w[m-n]e^{-j\lambda m}. \tag{10.25}$$

The relationship between the definitions of Eqs. (10.18) and (10.25) is easily shown to be

$$\check{X}[n, \lambda) = e^{-j\lambda n} X[n, \lambda) \tag{10.26}$$

The definition of Eq. (10.18) is particularly convenient when we consider using the DFT to obtain samples in λ of the time-dependent Fourier transform, since, if $w[m]$ is of finite length in the range $0 \leq m \leq (L-1)$, then so is $x[n+m]w[m]$. On the other hand, the definition of Eq. (10.25) has some advantages for the interpretation of Fourier analysis in terms of filter banks. Since our primary interest is in applications of the DFT, we will base our discussion on Eq. (10.18).

10.3.1 The Effect of the Window

The primary purpose of the window in the time-dependent Fourier transform is to limit the extent of the sequence to be transformed so that the spectral characteristics are reasonably stationary over the duration of the window. The more rapidly the signal characteristics change, the shorter the window should be. We saw in Section 10.2 that as the window becomes shorter, frequency resolution decreases. The same effect is true, of course, for $X[n, \lambda)$. On the other hand, as the window length decreases, the ability to resolve changes with time increases. Consequently, the choice of window length becomes a trade-off between frequency resolution and time resolution.

The effect of the window on the properties of the time-dependent Fourier transform can be seen by assuming that the signal $x[n]$ has a conventional discrete-time Fourier transform $X(e^{j\omega})$. First let us assume that the window is unity for all m; i.e., assume that there is no window at all. Then, from Eq. (10.18),

$$X[n, \lambda) = X(e^{j\lambda})e^{j\lambda n}. \tag{10.27}$$

Of course, a typical window for spectrum analysis tapers to zero so as to select only a portion of the signal for analysis. As discussed in Section 10.2, the length and shape of the window are chosen so that the Fourier transform of the window is narrow in frequency compared with variations with frequency of the Fourier transform of the signal. The Fourier transform of a typical window is illustrated in Figure 10.13(a).

If we consider the time-dependent Fourier transform for fixed n, then it follows from the properties of Fourier transforms that

$$X[n, \lambda) = \frac{1}{2\pi} \int_0^{2\pi} e^{j\theta n} X(e^{j\theta}) W(e^{j(\lambda-\theta)}) d\theta; \tag{10.28}$$

i.e., the Fourier transform of the shifted signal is convolved with the Fourier transform of the window. This is similar to Eq. (10.2), except that in Eq. (10.2) we assumed that the signal was not successively shifted relative to the window. Here we compute a Fourier transform for each value of n. In Section 10.2 we saw that the ability to resolve two narrowband signal components depends on the width of the main lobe of the Fourier