8

THE DISCRETE FOURIER TRANSFORM

8.0 INTRODUCTION

In Chapters 2 and 3 we discussed the representation of sequences and linear time-invariant systems in terms of the Fourier and z-transforms, respectively. For finite-duration sequences, it is possible to develop an alternative Fourier representation, referred to as the *discrete Fourier transform* (DFT). The DFT is itself a sequence rather than a function of a continuous variable, and it corresponds to samples, equally spaced in frequency, of the Fourier transform of the signal. In addition to its theoretical importance as a Fourier representation of sequences, the DFT plays a central role in the implementation of a variety of digital signal-processing algorithms. This is because efficient algorithms exist for the computation of the DFT. These algorithms will be discussed in detail in Chapter 9. The application of the DFT to spectral analysis will be described in Chapter 10.

Although several points of view can be taken toward the derivation and interpretation of the DFT representation of a finite-duration sequence, we have chosen to base our presentation on the relationship between periodic sequences and finite-length sequences. We will begin by considering the Fourier series representation of periodic sequences. While this representation is important in its own right, we are most often interested in the application of Fourier series results to the representation of finite-length sequences. We accomplish this by constructing a periodic sequence for which each period is identical to the finite-length sequence. As we will see, the Fourier series representation of the periodic sequence corresponds to the DFT of the finite-length sequence. Thus, our approach is to define the Fourier series representation for periodic sequences and to study the properties of such representations. Then we repeat essentially the same derivations, assuming that the sequence to be represented is a finite-length sequence. This approach to the DFT emphasizes the fundamental inherent periodicity of the DFT representation and ensures that this periodicity is not overlooked in applications of the DFT.

8.1 REPRESENTATION OF PERIODIC SEQUENCES: THE DISCRETE FOURIER SERIES

Consider a sequence $\tilde{x}[n]$ that is periodic[1] with period N, so that $\tilde{x}[n] = \tilde{x}[n + rN]$ for any integer values of n and r. As with continuous-time periodic signals, such a sequence can be represented by a Fourier series corresponding to a sum of harmonically related complex exponential sequences, i.e., complex exponentials with frequencies that are integer multiples of the fundamental frequency $(2\pi/N)$ associated with the periodic sequence $\tilde{x}[n]$. These periodic complex exponentials are of the form

$$e_k[n] = e^{j(2\pi/N)kn} = e_k[n + rN], \tag{8.1}$$

where k is an integer, and the Fourier series representation then has the form[2]

$$\tilde{x}[n] = \frac{1}{N}\sum_{k} \tilde{X}[k]e^{j(2\pi/N)kn}. \tag{8.2}$$

The Fourier series representation of a continuous-time periodic signal generally requires infinitely many harmonically related complex exponentials, whereas the Fourier series for any discrete-time signal with period N requires only N harmonically related complex exponentials. To see this, note that the harmonically related complex exponentials $e_k[n]$ in Eq. (8.1) are identical for values of k separated by N; i.e., $e_0[n] = e_N[n]$, $e_1[n] = e_{N+1}[n]$, and, in general,

$$e_{k+\ell N}[n] = e^{j(2\pi/N)(k+\ell N)n} = e^{j(2\pi/N)kn}e^{j2\pi\ell n} = e^{j(2\pi/N)kn} = e_k[n], \tag{8.3}$$

where ℓ is an integer. Consequently, the set of N periodic complex exponentials $e_0[n]$, $e_1[n], \ldots, e_{N-1}[n]$ defines all the distinct periodic complex exponentials with frequencies that are integer multiples of $(2\pi/N)$. Thus, the Fourier series representation of a periodic sequence $\tilde{x}[n]$ need contain only N of these complex exponentials, and hence, it has the form

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}[k]e^{j(2\pi/N)kn}. \tag{8.4}$$

To obtain the sequence of Fourier series coefficients $\tilde{X}[k]$ from the periodic sequence $\tilde{x}[n]$, we exploit the orthogonality of the set of complex exponential sequences. After multiplying both sides of Eq. (8.4) by $e^{-j(2\pi/N)rn}$ and summing from $n = 0$ to $n = N - 1$, we obtain

$$\sum_{n=0}^{N-1} \tilde{x}[n]e^{-j(2\pi/N)rn} = \sum_{n=0}^{N-1} \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}[k]e^{j(2\pi/N)(k-r)n}. \tag{8.5}$$

[1]Henceforth, we will use the tilde (˜) to denote periodic sequences whenever it is important to clearly distinguish between periodic and aperiodic sequences.

[2]The multiplicative constant $1/N$ is included in Eq. (8.2) for convenience. It could also be absorbed into the definition of $\tilde{X}[k]$.

After interchanging the order of summation on the right-hand side, we see that Eq. (8.5) becomes

$$\sum_{n=0}^{N-1} \tilde{x}[n]e^{-j(2\pi/N)rn} = \sum_{k=0}^{N-1} \tilde{X}[k]\left[\frac{1}{N}\sum_{n=0}^{N-1} e^{j(2\pi/N)(k-r)n}\right]. \tag{8.6}$$

The following identity expresses the orthogonality of the complex exponentials:

$$\frac{1}{N}\sum_{n=0}^{N-1} e^{j(2\pi/N)(k-r)n} = \begin{cases} 1, & k-r=mN, \quad m \text{ an integer}, \\ 0, & \text{otherwise}. \end{cases} \tag{8.7}$$

This identity can easily be proved (see Problem 8.51), and when it is applied to the summation in brackets in Eq. (8.6), the result is

$$\sum_{n=0}^{N-1} \tilde{x}[n]e^{-j(2\pi/N)rn} = \tilde{X}[r]. \tag{8.8}$$

Thus, the Fourier series coefficients $\tilde{X}[k]$ in Eq. (8.4) are obtained from $\tilde{x}[n]$ by the relation

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n]e^{-j(2\pi/N)kn}. \tag{8.9}$$

Note that the sequence $\tilde{X}[k]$ is periodic with period N; i.e., $\tilde{X}[0] = \tilde{X}[N]$, $\tilde{X}[1] = \tilde{X}[N+1]$, and, more generally,

$$\tilde{X}[k+N] = \sum_{n=0}^{N-1} \tilde{x}[n]e^{-j(2\pi/N)(k+N)n}$$

$$= \left(\sum_{n=0}^{N-1} \tilde{x}[n]e^{-j(2\pi/N)kn}\right) e^{-j2\pi n} = \tilde{X}[k]$$

for any integer k.

The Fourier series coefficients can be interpreted to be a sequence of finite length, given by Eq. (8.9) for $k = 0, \ldots, (N-1)$, and zero otherwise, or as a periodic sequence defined for all k by Eq. (8.9). Clearly, both of these interpretations are acceptable, since in Eq. (8.4) we use only the values of $\tilde{X}[k]$ for $0 \le k \le (N-1)$. An advantage of interpreting the Fourier series coefficients $\tilde{X}[k]$ as a periodic sequence is that there is then a duality between the time and frequency domains for the Fourier series representation of periodic sequences. Equations (8.9) and (8.4) together are an analysis–synthesis pair and will be referred to as the *discrete Fourier series* (DFS) representation of a periodic sequence. For convenience in notation, these equations are often written in terms of the complex quantity

$$W_N = e^{-j(2\pi/N)}. \tag{8.10}$$

With this notation, the DFS analysis–synthesis pair is expressed as follows:

$$\text{Analysis equation:} \quad \tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n]W_N^{kn}. \tag{8.11}$$

$$\text{Synthesis equation:} \quad \tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}[k]W_N^{-kn}. \tag{8.12}$$

In both of these equations, $\tilde{X}[k]$ and $\tilde{x}[n]$ are periodic sequences. We will sometimes find it convenient to use the notation

$$\tilde{x}[n] \stackrel{\mathcal{DFS}}{\longleftrightarrow} \tilde{X}[k] \tag{8.13}$$

to signify the relationships of Eqs. (8.11) and (8.12). The following examples illustrate the use of those equations.

Example 8.1 Discrete Fourier Series of a Periodic Impulse Train

We consider the periodic impulse train

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} \delta[n - rN] = \begin{cases} 1, & n = rN, \quad r \text{ any integer}, \\ 0, & \text{otherwise}. \end{cases} \tag{8.14}$$

Since $\tilde{x}[n] = \delta[n]$ for $0 \le n \le N - 1$, the DFS coefficients are found, using Eq. (8.11), to be

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \delta[n] W_N^{kn} = W_N^0 = 1. \tag{8.15}$$

In this case, $\tilde{X}[k]$ is the same for all k. Thus, substituting Eq. (8.15) into Eq. (8.12) leads to the representation

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} \delta[n - rN] = \frac{1}{N} \sum_{k=0}^{N-1} W_N^{-kn} = \frac{1}{N} \sum_{k=0}^{N-1} e^{j(2\pi/N)kn}. \tag{8.16}$$

(Note the similarity to the orthogonality relation of Eq. (8.7).)

Example 8.1 produced a useful representation of a periodic impulse train in terms of a sum of complex exponentials, where all the complex exponentials have the same magnitude and phase and add to unity at integer multiples of N and to zero for all other integers. If we look closely at Eqs. (8.11) and (8.12), we see that the two equations are very similar, differing only in a constant multiplier and the sign of the exponents. This duality between the periodic sequence $\tilde{x}[n]$ and its discrete Fourier series coefficients $\tilde{X}[k]$ is illustrated in the following example.

Example 8.2 Duality in the Discrete Fourier Series

Here we let the discrete Fourier series coefficients be the periodic impulse train

$$\tilde{Y}[k] = \sum_{r=-\infty}^{\infty} N\delta[k - rN].$$

Substituting $\tilde{Y}[k]$ into Eq. (8.12) gives

$$\tilde{y}[n] = \frac{1}{N} \sum_{k=0}^{N-1} N\delta[k] W_N^{-kn} = W_N^{-0} = 1.$$

In this case, $\tilde{y}[n] = 1$ for all n. Comparing this result with the results for $\tilde{x}[n]$ and $\tilde{X}[k]$ of Example 8.1, we see that $\tilde{Y}[k] = N\tilde{x}[k]$ and $\tilde{y}[n] = \tilde{X}[n]$. In Section 8.2.3, we will show that this example is a special case of a more general duality property.

If the sequence $\tilde{x}[n]$ is equal to unity over only part of one period, we can also obtain a closed-form expression for the DFS coefficients. This is illustrated by the following example.

Example 8.3 The Discrete Fourier Series of a Periodic Rectangular Pulse Train

For this example, $\tilde{x}[n]$ is the sequence shown in Figure 8.1, whose period is $N = 10$.

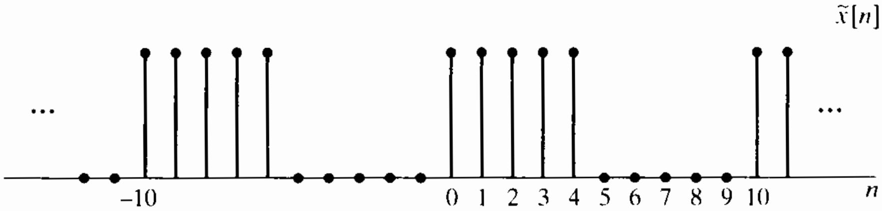

Figure 8.1 Periodic sequence with period $N = 10$ for which the Fourier series representation is to be computed.

From Eq. (8.11),

$$\tilde{X}[k] = \sum_{n=0}^{4} W_{10}^{kn} = \sum_{n=0}^{4} e^{-j(2\pi/10)kn}. \tag{8.17}$$

This finite sum has the closed form

$$\tilde{X}[k] = \frac{1 - W_{10}^{5k}}{1 - W_{10}^{k}} = e^{-j(4\pi k/10)} \frac{\sin(\pi k/2)}{\sin(\pi k/10)}. \tag{8.18}$$

The magnitude and phase of the periodic sequence $\tilde{X}[k]$ are shown in Figure 8.2.

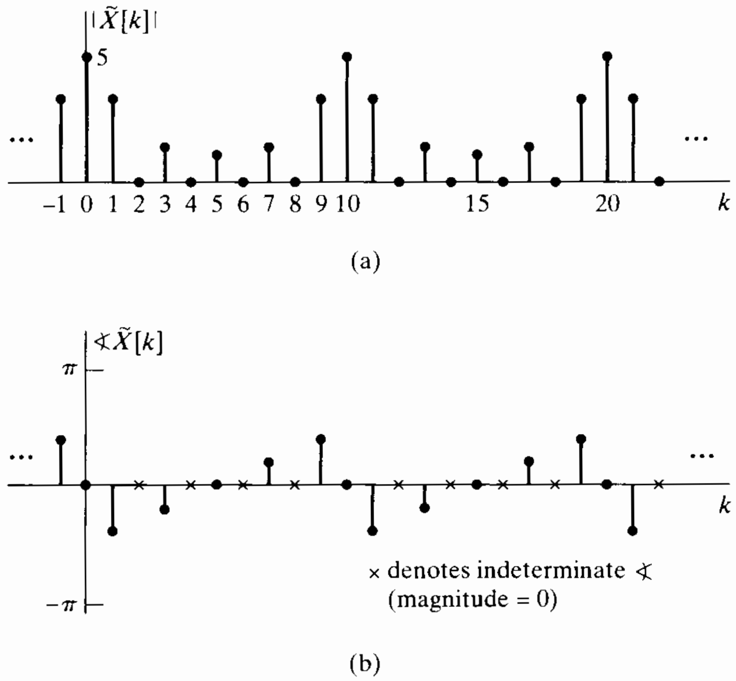

Figure 8.2 Magnitude and phase of the Fourier series coefficients of the sequence of Figure 8.1.

We have shown that any periodic sequence can be represented as a sum of complex exponential sequences. The key results are summarized in Eqs. (8.11) and (8.12). As we will see, these relationships are the basis for the DFT, which focuses on finite-length sequences. Before discussing the DFT, however, we will consider some of the basic properties of the DFS representation of periodic sequences, and then we will show how we can use the DFS representation to obtain a Fourier transform representation of periodic signals.

8.2 PROPERTIES OF THE DISCRETE FOURIER SERIES

Just as with Fourier series and Fourier and Laplace transforms for continuous-time signals, and with z-transforms for discrete-time aperiodic sequences, certain properties of discrete Fourier series are of fundamental importance to its successful use in signal-processing problems. In this section, we summarize these important properties. It is not surprising that many of the basic properties are analogous to properties of the z-transform and Fourier transform. However, we will be careful to point out where the periodicity of both $\tilde{x}[n]$ and $\tilde{X}[k]$ results in some important distinctions. Furthermore, an exact duality exists between the time and frequency domains in the DFS representation that does not exist in the Fourier transform and z-transform representation of sequences.

8.2.1 Linearity

Consider two periodic sequences $\tilde{x}_1[n]$ and $\tilde{x}_2[n]$, both with period N, such that

$$\tilde{x}_1[n] \overset{\mathcal{DFS}}{\longleftrightarrow} \tilde{X}_1[k] \tag{8.19a}$$

and

$$\tilde{x}_2[n] \overset{\mathcal{DFS}}{\longleftrightarrow} \tilde{X}_2[k] \tag{8.19b}$$

Then

$$a\tilde{x}_1[n] + b\tilde{x}_2[n] \overset{\mathcal{DFS}}{\longleftrightarrow} a\tilde{X}_1[k] + b\tilde{X}_2[k]. \tag{8.20}$$

This linearity property follows immediately from the form of Eqs. (8.11) and (8.12).

8.2.2 Shift of a Sequence

If a periodic sequence $\tilde{x}[n]$ has Fourier coefficients $\tilde{X}[k]$, then $\tilde{x}[n-m]$ is a shifted version of $\tilde{x}[n]$, and

$$\tilde{x}[n-m] \overset{\mathcal{DFS}}{\longleftrightarrow} W_N^{km}\tilde{X}[k]. \tag{8.21}$$

The proof of this property is considered in Problem 8.52. Any shift that is greater than or equal to the period (i.e., $m \geq N$) cannot be distinguished in the time domain from a shorter shift m_1 such that $m = m_1 + m_2 N$, where m_1 and m_2 are integers and $0 \leq m_1 \leq N-1$. (Another way of stating this is that $m_1 = m$ modulo N or, equivalently, m_1 is the remainder when m is divided by N.) It is easily shown that with this representation of m, $W_N^{km} = W_N^{km_1}$; i.e., as it must be, the ambiguity of the shift in the time domain is also manifest in the frequency-domain representation.

Because the sequence of Fourier series coefficients of a periodic sequence is a periodic sequence, a similar result applies to a shift in the Fourier coefficients by an integer ℓ. Specifically,

$$W_N^{-n\ell}\tilde{x}[n] \overset{\mathcal{DFS}}{\longleftrightarrow} \tilde{X}[k-\ell]. \tag{8.22}$$

Note the difference in the sign of the exponents in Eqs. (8.21) and (8.22).

8.2.3 Duality

Because of the strong similarity between the Fourier analysis and synthesis equations in continuous time, there is a duality between the time domain and frequency domain. However, for the discrete-time Fourier transform of aperiodic signals, no similar duality exists, since aperiodic signals and their Fourier transforms are very different kinds of functions: Aperiodic discrete-time signals are, of course, aperiodic sequences, while their Fourier transforms are always periodic functions of a continuous frequency variable.

From Eqs. (8.11) and (8.12), we see that the DFS analysis and synthesis equations differ only in a factor of $1/N$ and in the sign of the exponent of W_N. Furthermore, a periodic sequence and its DFS coefficients are the same kinds of functions; they are both periodic sequences. Specifically, taking account of the factor $1/N$ and the difference in sign in the exponent between Eqs. (8.11) and (8.12), it follows from Eq. (8.12) that

$$N\tilde{x}[-n] = \sum_{k=0}^{N-1} \tilde{X}[k]W_N^{kn} \tag{8.23}$$

or, interchanging the roles of n and k in Eq. (8.23),

$$N\tilde{x}[-k] = \sum_{n=0}^{N-1} \tilde{X}[n]W_N^{nk}. \tag{8.24}$$

We see that Eq. (8.24) is similar to Eq. (8.11). In other words, the sequence of DFS coefficients of the periodic sequence $\tilde{X}[n]$ is $N\tilde{x}[-k]$, i.e., the original periodic sequence in reverse order and multiplied by N. This duality property is summarized as follows: If

$$\tilde{x}[n] \overset{\mathcal{DFS}}{\longleftrightarrow} \tilde{X}[k], \tag{8.25a}$$

then

$$\tilde{X}[n] \overset{\mathcal{DFS}}{\longleftrightarrow} N\tilde{x}[-k]. \tag{8.25b}$$

8.2.4 Symmetry Properties

As we discussed in Section 2.8, the Fourier transform of an aperiodic sequence has a number of useful symmetry properties. The same basic properties also hold for the DFS representation of a periodic sequence. The derivation of these properties, which is similar in style to the derivations in Chapter 2, is left as an exercise. (See Problem 8.53.) The resulting properties are summarized for reference as properties 9–17 in Table 8.1 in Section 8.2.6.

8.2.5 Periodic Convolution

Let $\tilde{x}_1[n]$ and $\tilde{x}_2[n]$ be two periodic sequences, each with period N and with discrete Fourier series coefficients denoted by $\tilde{X}_1[k]$ and $\tilde{X}_2[k]$, respectively. If we form the product

$$\tilde{X}_3[k] = \tilde{X}_1[k]\tilde{X}_2[k], \tag{8.26}$$

then the periodic sequence $\tilde{x}_3[n]$ with Fourier series coefficients $\tilde{X}_3[k]$ is

$$\tilde{x}_3[n] = \sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n-m]. \tag{8.27}$$

This result is not surprising, since our previous experience with transforms suggests that multiplication of frequency-domain functions corresponds to convolution of time-domain functions and Eq. (8.27) looks very much like a convolution sum. Equation (8.27) involves the summation of values of the product of $\tilde{x}_1[m]$ with $\tilde{x}_2[n-m]$, which is a time-reversed and time-shifted version of $\tilde{x}_2[m]$, just as in aperiodic discrete convolution. However, the sequences in Eq. (8.27) are all periodic with period N, and the summation is over only one period. A convolution in the form of Eq. (8.27) is referred to as a *periodic convolution*. Just as with aperiodic convolution, periodic convolution is commutative; i.e.,

$$\tilde{x}_3[n] = \sum_{m=0}^{N-1} \tilde{x}_2[m]\tilde{x}_1[n-m]. \tag{8.28}$$

To demonstrate that $\tilde{X}_3[k]$, given by Eq. (8.26), is the sequence of Fourier coefficients corresponding to $\tilde{x}_3[n]$ given by Eq. (8.27), let us first apply the DFS analysis equation (8.11) to Eq. (8.27) to obtain

$$\tilde{X}_3[k] = \sum_{n=0}^{N-1} \left(\sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n-m] \right) W_N^{kn}, \tag{8.29}$$

which, after we interchange the order of summation, becomes

$$\tilde{X}_3[k] = \sum_{m=0}^{N-1} \tilde{x}_1[m] \left(\sum_{n=0}^{N-1} \tilde{x}_2[n-m] W_N^{kn} \right). \tag{8.30}$$

The inner sum on the index n is the DFS for the shifted sequence $\tilde{x}_2[n-m]$. Therefore, from the shifting property of Section 8.2.2, we obtain

$$\sum_{n=0}^{N-1} \tilde{x}_2[n-m] W_N^{kn} = W_N^{km} \tilde{X}_2[k],$$

which can be substituted into Eq. (8.30) to yield

$$\tilde{X}_3[k] = \sum_{m=0}^{N-1} \tilde{x}_1[m] W_N^{km} \tilde{X}_2[k] = \left(\sum_{m=0}^{N-1} \tilde{x}_1[m] W_N^{km} \right) \tilde{X}_2[k] = \tilde{X}_1[k]\tilde{X}_2[k]. \tag{8.31}$$

In summary,

$$\sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n-m] \stackrel{\mathcal{DFS}}{\longleftrightarrow} \tilde{X}_1[k]\tilde{X}_2[k]. \tag{8.32}$$

The periodic convolution of periodic sequences thus corresponds to multiplication of the corresponding periodic sequences of Fourier series coefficients.

Since periodic convolutions are somewhat different from aperiodic convolutions, it is worthwhile to consider the mechanics of evaluating Eq. (8.27). First note that Eq. (8.27) calls for the product of sequences $\tilde{x}_1[m]$ and $\tilde{x}_2[n-m] = \tilde{x}_2[-(m-n)]$ viewed as functions of m with n fixed. This is the same as for an aperiodic convolution, but with the following two major differences:

1. The sum is over the finite interval $0 \le m \le N-1$.
2. The values of $\tilde{x}_2[n-m]$ in the interval $0 \le m \le N-1$ repeat periodically for m outside of that interval.

These details are illustrated by the following example.

Example 8.4 Periodic Convolution

An illustration of the procedure for forming the periodic convolution of two periodic sequences corresponding to Eq. (8.27) is given in Figure 8.3, where we have illustrated

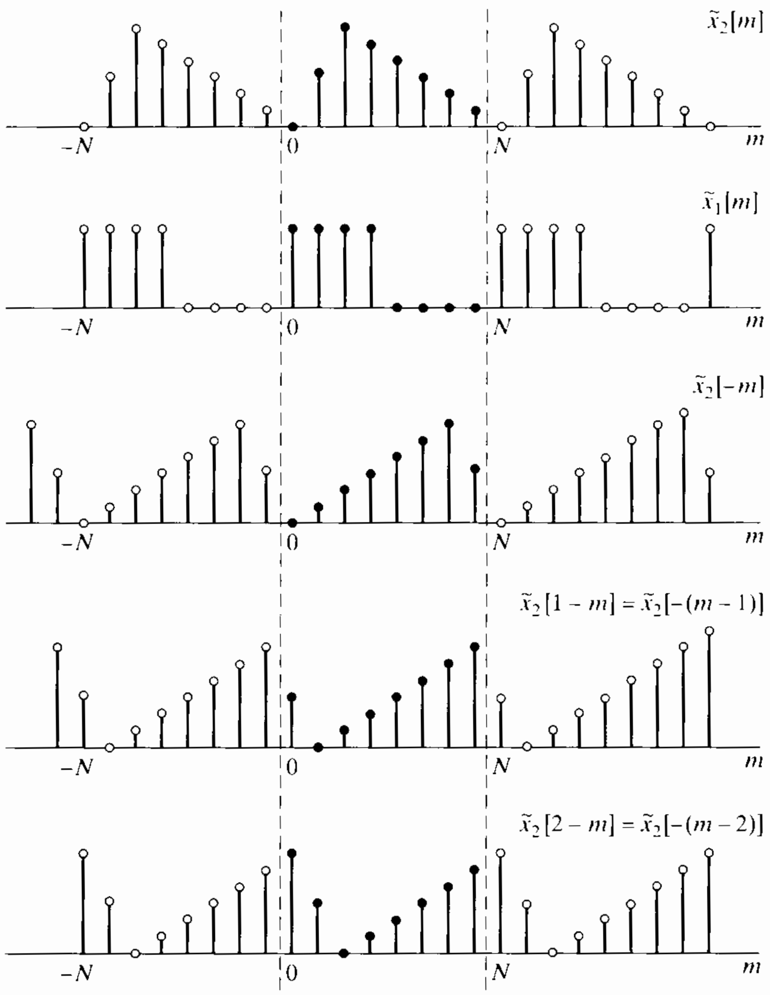

Figure 8.3 Procedure for forming the periodic convolution of two periodic sequences.

the sequences $\tilde{x}_2[m]$, $\tilde{x}_1[m]$, $\tilde{x}_2[-m]$, $\tilde{x}_2[1-m] = \tilde{x}_2[-(m-1)]$, and $\tilde{x}_2[2-m] = \tilde{x}_2[-(m-2)]$. To evaluate $\tilde{x}_3[n]$ in Eq. (8.27) for $n = 2$, for example, we multiply $\tilde{x}_1[m]$ by $\tilde{x}_2[2-m]$ and then sum the product terms $\tilde{x}_1[m]\tilde{x}_2[2-m]$ for $0 \le m \le N-1$, obtaining $\tilde{x}_3[2]$. As n changes, the sequence $\tilde{x}_2[n-m]$ shifts appropriately, and Eq. (8.27) is evaluated for each value of $0 \le n \le N-1$. Note that as the sequence $\tilde{x}_2[n-m]$ shifts to the right or left, values that leave the interval between the dotted lines at one end reappear at the other end because of the periodicity. Because of the periodicity of $\tilde{x}_3[n]$, there is no need to continue to evaluate Eq. (8.27) outside the interval $0 \le n \le N-1$.

The duality theorem (Section 8.2.3) suggests that if the roles of time and frequency are interchanged, we will obtain a result almost identical to the previous result. That is, the periodic sequence

$$\tilde{x}_3[n] = \tilde{x}_1[n]\tilde{x}_2[n], \tag{8.33}$$

where $\tilde{x}_1[n]$ and $\tilde{x}_2[n]$ are periodic sequences, each with period N, has the discrete Fourier series coefficients given by

$$\tilde{X}_3[k] = \frac{1}{N}\sum_{\ell=0}^{N-1} \tilde{X}_1[\ell]\tilde{X}_2[k-\ell], \tag{8.34}$$

corresponding to $1/N$ times the periodic convolution of $\tilde{X}_1[k]$ and $\tilde{X}_2[k]$. This result can also be verified by substituting $\tilde{X}_3[k]$, given by Eq. (8.34), into the Fourier series relation of Eq. (8.12) to obtain $\tilde{x}_3[n]$.

8.2.6 Summary of Properties of the DFS Representation of Periodic Sequences

The properties of the discrete Fourier series representation discussed in this section are summarized in Table 8.1.

TABLE 8.1 SUMMARY OF PROPERTIES OF THE DFS

	Periodic Sequence (Period N)	DFS Coefficients (Period N)
1.	$\tilde{x}[n]$	$\tilde{X}[k]$ periodic with period N
2.	$\tilde{x}_1[n]$, $\tilde{x}_2[n]$	$\tilde{X}_1[k]$, $\tilde{X}_2[k]$ periodic with period N
3.	$a\tilde{x}_1[n] + b\tilde{x}_2[n]$	$a\tilde{X}_1[k] + b\tilde{X}_2[k]$
4.	$\tilde{X}[n]$	$N\tilde{x}[-k]$
5.	$\tilde{x}[n-m]$	$W_N^{km}\tilde{X}[k]$
6.	$W_N^{-\ell n}\tilde{x}[n]$	$\tilde{X}[k-\ell]$
7.	$\sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n-m]$ (periodic convolution)	$\tilde{X}_1[k]\tilde{X}_2[k]$
8.	$\tilde{x}_1[n]\tilde{x}_2[n]$	$\frac{1}{N}\sum_{\ell=0}^{N-1} \tilde{X}_1[\ell]\tilde{X}_2[k-\ell]$ (periodic convolution)
9.	$\tilde{x}^*[n]$	$\tilde{X}^*[-k]$

(continued)

TABLE 8.1 (*Continued*)

Periodic Sequence (Period N)	DFS Coefficients (Period N)
10. $\tilde{x}^*[-n]$	$\tilde{X}^*[k]$
11. $\mathcal{R}e\{\tilde{x}[n]\}$	$\tilde{X}_e[k] = \frac{1}{2}(\tilde{X}[k] + \tilde{X}^*[-k])$
12. $j\mathcal{J}m\{\tilde{x}[n]\}$	$\tilde{X}_o[k] = \frac{1}{2}(\tilde{X}[k] - \tilde{X}^*[-k])$
13. $\tilde{x}_e[n] = \frac{1}{2}(\tilde{x}[n] + \tilde{x}^*[-n])$	$\mathcal{R}e\{\tilde{X}[k]\}$
14. $\tilde{x}_o[n] = \frac{1}{2}(\tilde{x}[n] - \tilde{x}^*[-n])$	$j\mathcal{J}m\{\tilde{X}[k]\}$
Properties 15–17 apply only when $x[n]$ is real.	
15. Symmetry properties for $\tilde{x}[n]$ real.	$\begin{cases} \tilde{X}[k] = \tilde{X}^*[-k] \\ \mathcal{R}e\{\tilde{X}[k]\} = \mathcal{R}e\{\tilde{X}[-k]\} \\ \mathcal{J}m\{\tilde{X}[k]\} = -\mathcal{J}m\{\tilde{X}[-k]\} \\ \lvert\tilde{X}[k]\rvert = \lvert\tilde{X}[-k]\rvert \\ \sphericalangle\tilde{X}[k] = -\sphericalangle\tilde{X}[-k] \end{cases}$
16. $\tilde{x}_e[n] = \frac{1}{2}(\tilde{x}[n] + \tilde{x}[-n])$	$\mathcal{R}e\{\tilde{X}[k]\}$
17. $\tilde{x}_0[n] = \frac{1}{2}(\tilde{x}[n] - \tilde{x}[-n])$	$j\mathcal{J}m\{\tilde{X}[k]\}$

8.3 THE FOURIER TRANSFORM OF PERIODIC SIGNALS

As discussed in Section 2.7, uniform convergence of the Fourier transform of a sequence requires that the sequence be absolutely summable, and mean-square convergence requires that the sequence be square summable. Periodic sequences satisfy neither condition, because they do not approach zero as n approaches $\pm\infty$. However, as we discussed briefly in Section 2.7, sequences that can be expressed as a sum of complex exponentials can be considered to have a Fourier transform representation in the form of Eq. (2.152), i.e., as a train of impulses. Similarly, it is often useful to incorporate the discrete Fourier series representation of periodic signals within the framework of the Fourier transform. This can be done by interpreting the Fourier transform of a periodic signal to be an impulse train in the frequency domain with the impulse values proportional to the DFS coefficients for the sequence. Specifically, if $\tilde{x}[n]$ is periodic with period N and the corresponding discrete Fourier series coefficients are $\tilde{X}[k]$, then the Fourier transform of $\tilde{x}[n]$ is defined to be the impulse train

$$\tilde{X}(e^{j\omega}) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{N}\tilde{X}[k]\delta\left(\omega - \frac{2\pi k}{N}\right). \tag{8.35}$$

Note that $\tilde{X}(e^{j\omega})$ has the necessary periodicity with period 2π since $\tilde{X}[k]$ is periodic with period N and the impulses are spaced at integer multiples of $2\pi/N$, where N is an integer. To show that $\tilde{X}(e^{j\omega})$ as defined in Eq. (8.35) is a Fourier transform representation of the periodic sequence $\tilde{x}[n]$, we substitute Eq. (8.35) into the inverse Fourier transform equation (2.133); i.e.,

$$\frac{1}{2\pi}\int_{0-\epsilon}^{2\pi-\epsilon} \tilde{X}(e^{j\omega})e^{j\omega n}d\omega = \frac{1}{2\pi}\int_{0-\epsilon}^{2\pi-\epsilon} \sum_{k=-\infty}^{\infty} \frac{2\pi}{N}\tilde{X}[k]\delta\left(\omega - \frac{2\pi k}{N}\right)e^{j\omega n}d\omega, \tag{8.36}$$

where ϵ satisfies the inequality $0 < \epsilon < (2\pi/N)$. Recall that in evaluating the inverse Fourier transform, we can integrate over *any* interval of length 2π, since the integrand

$\tilde{X}(e^{j\omega})e^{j\omega n}$ is periodic with period 2π. In Eq. (8.36) the integration limits are denoted $0-\epsilon$ and $2\pi-\epsilon$, which means that the integration is from just before $\omega = 0$ to just before $\omega = 2\pi$. These limits are convenient because they include the impulse at $\omega = 0$ and exclude the impulse at $\omega = 2\pi$. Interchanging the order of integration and summation leads to

$$\begin{aligned}\frac{1}{2\pi}\int_{0-\epsilon}^{2\pi-\epsilon} \tilde{X}(e^{j\omega})e^{j\omega n}d\omega &= \frac{1}{N}\sum_{k=-\infty}^{\infty}\tilde{X}[k]\int_{0-\epsilon}^{2\pi-\epsilon}\delta\left(\omega-\frac{2\pi k}{N}\right)e^{j\omega n}d\omega \\ &= \frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]e^{j(2\pi/N)kn}.\end{aligned} \tag{8.37}$$

The final form of Eq. (8.37) results because only the impulses corresponding to $k = 0, 1, \ldots, (N-1)$ are included in the interval between $\omega = 0-\epsilon$ and $\omega = 2\pi-\epsilon$.

Comparing Eq. (8.37) and Eq. (8.12), we see that the final right-hand side of Eq. (8.37) is exactly equal to the Fourier series representation for $\tilde{x}[n]$, as specified by Eq. (8.12). Consequently, the inverse Fourier transform of the impulse train in Eq. (8.35) is the periodic signal $\tilde{x}[n]$, as desired.

Although the Fourier transform of a periodic sequence does not converge in the normal sense, the introduction of impulses permits us to include periodic sequences formally within the framework of Fourier transform analysis. This approach was also used in Chapter 2 to obtain a Fourier transform representation of other nonsummable sequences, such as the two-sided constant sequence (Example 2.23) or the complex exponential sequence (Example 2.24). Although the discrete Fourier series representation is adequate for most purposes, the Fourier transform representation of Eq. (8.35) sometimes leads to simpler or more compact expressions and simplified analysis.

Example 8.5 The Fourier Transform of a Periodic Impulse Train

Consider the periodic impulse train

$$\tilde{p}[n] = \sum_{r=-\infty}^{\infty}\delta[n-rN], \tag{8.38}$$

which is the same as the periodic sequence $\tilde{x}[n]$ considered in Example 8.1. From the results of that example, it follows that

$$\tilde{P}[k] = 1, \qquad \text{for all } k. \tag{8.39}$$

Therefore, the Fourier transform of $\tilde{p}[n]$ is

$$\tilde{P}(e^{j\omega}) = \sum_{k=-\infty}^{\infty}\frac{2\pi}{N}\delta\left(\omega-\frac{2\pi k}{N}\right). \tag{8.40}$$

The result of Example 8.5 is the basis for a useful interpretation of the relation between a periodic signal and a finite-length signal. Consider a finite-length signal $x[n]$

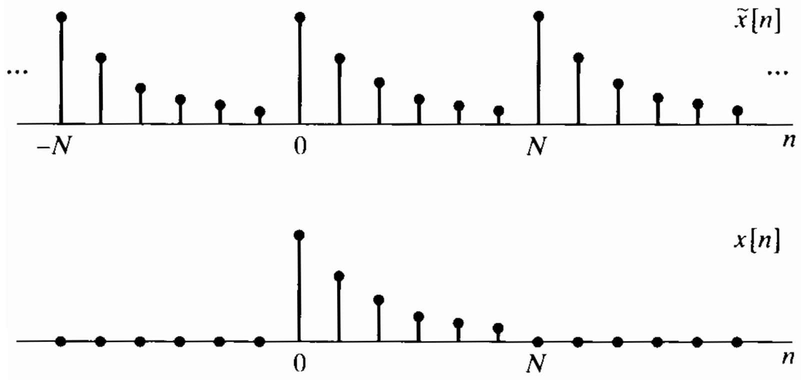

Figure 8.4 Periodic sequence $\tilde{x}[n]$ formed by repeating a finite-length sequence, $x[n]$, periodically. Alternatively, $x[n] = \tilde{x}[n]$ over one period and is zero otherwise.

such that $x[n] = 0$ except in the interval $0 \le n \le N - 1$, and consider the convolution of $x[n]$ with the periodic impulse train $\tilde{p}[n]$ of Example 8.5:

$$\begin{aligned}\tilde{x}[n] = x[n] * \tilde{p}[n] &= x[n] * \sum_{r=-\infty}^{\infty} \delta[n - rN] \\ &= \sum_{r=-\infty}^{\infty} x[n - rN].\end{aligned} \tag{8.41}$$

Equation (8.41) states that $\tilde{x}[n]$ consists of a set of periodically repeated copies of the finite-length sequence $x[n]$. Figure 8.4 illustrates how a periodic sequence $\tilde{x}[n]$ can be formed from a finite-length sequence $x[n]$ through Eq. (8.41). The Fourier transform of $x[n]$ is $X(e^{j\omega})$, and the Fourier transform of $\tilde{x}[n]$ is

$$\begin{aligned}\tilde{X}(e^{j\omega}) &= X(e^{j\omega})\tilde{P}(e^{j\omega}) \\ &= X(e^{j\omega}) \sum_{k=-\infty}^{\infty} \frac{2\pi}{N}\delta\left(\omega - \frac{2\pi k}{N}\right) \\ &= \sum_{k=-\infty}^{\infty} \frac{2\pi}{N} X(e^{j(2\pi/N)k})\delta\left(\omega - \frac{2\pi k}{N}\right).\end{aligned} \tag{8.42}$$

Comparing Eq. (8.42) with Eq. (8.35), we conclude that

$$\tilde{X}[k] = X(e^{j(2\pi/N)k}) = X(e^{j\omega})\big|_{\omega=(2\pi/N)k}. \tag{8.43}$$

In other words, the periodic sequence $\tilde{X}[k]$ of DFS coefficients in Eq. (8.11) has an interpretation as equally spaced samples of the Fourier transform of the finite-length sequence obtained by extracting one period of $\tilde{x}[n]$; i.e.,

$$x[n] = \begin{cases} \tilde{x}[n], & 0 \le n \le N - 1, \\ 0, & \text{otherwise}. \end{cases} \tag{8.44}$$

This is also consistent with Figure 8.4, where it is clear that $x[n]$ can be obtained from $\tilde{x}[n]$ using Eq. (8.44). We can verify Eq. (8.43) in yet another way. Since $x[n] = \tilde{x}[n]$ for

$0 \le n \le N-1$ and $x[n] = 0$ otherwise,

$$X(e^{j\omega}) = \sum_{n=0}^{N-1} x[n]e^{-j\omega n} = \sum_{n=0}^{N-1} \tilde{x}[n]e^{-j\omega n}. \tag{8.45}$$

Comparing Eq. (8.45) and Eq. (8.11), we see again that

$$\tilde{X}[k] = X(e^{j\omega})|_{\omega=2\pi k/N}. \tag{8.46}$$

This corresponds to sampling the Fourier transform at N equally spaced frequencies between $\omega = 0$ and $\omega = 2\pi$ with a frequency spacing of $2\pi/N$.

Example 8.6 Relationship Between the Fourier Series Coefficients and the Fourier Transform of One Period

We again consider the sequence $\tilde{x}[n]$ of Example 8.3, which is shown in Figure 8.1. One period of $\tilde{x}[n]$ for the sequence in Figure 8.1 is

$$x[n] = \begin{cases} 1, & 0 \le n \le 4, \\ 0, & \text{otherwise.} \end{cases} \tag{8.47}$$

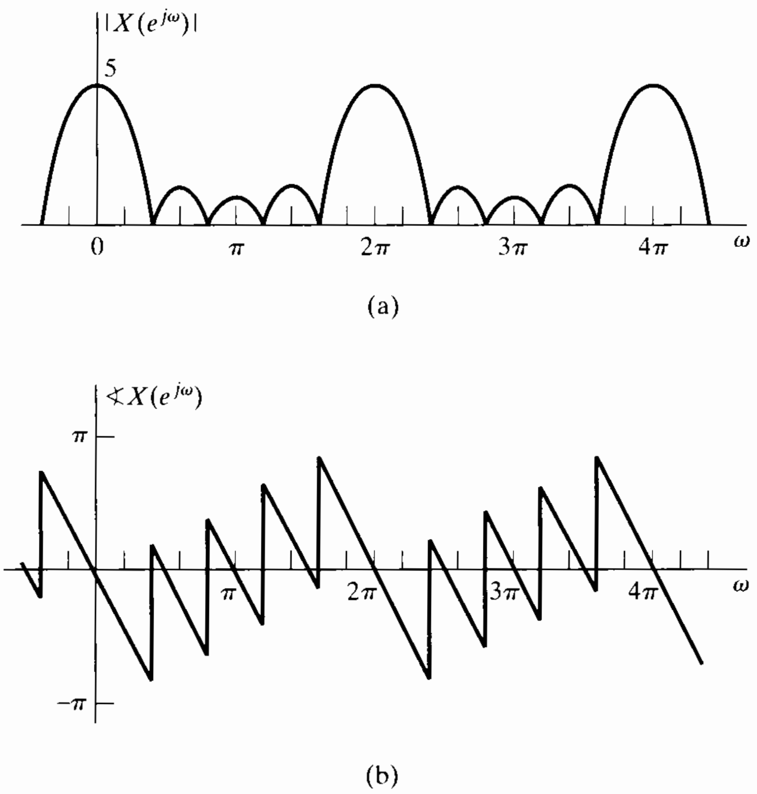

Figure 8.5 Magnitude and phase of the Fourier transform of one period of the sequence in Figure 8.1.

The Fourier transform of one period of $\tilde{x}[n]$ is given by

$$X(e^{j\omega}) = \sum_{n=0}^{4} e^{-j\omega n} = e^{-j2\omega} \frac{\sin(5\omega/2)}{\sin(\omega/2)}. \tag{8.48}$$

Equation (8.46) can be shown to be satisfied for this example by substituting $\omega = 2\pi k/10$ into Eq. (8.48), giving

$$\tilde{X}[k] = e^{-j(4\pi k/10)} \frac{\sin(\pi k/2)}{\sin(\pi k/10)},$$

which is identical to the result in Eq. (8.18). The magnitude and phase of $X(e^{j\omega})$ are sketched in Figure 8.5. Note that the phase is discontinuous at the frequencies where $X(e^{j\omega}) = 0$. That the sequences in Figures 8.2(a) and (b) correspond to samples of Figures 8.5(a) and (b), respectively, is demonstrated in Figure 8.6, where Figures 8.2 and 8.5 have been superimposed.

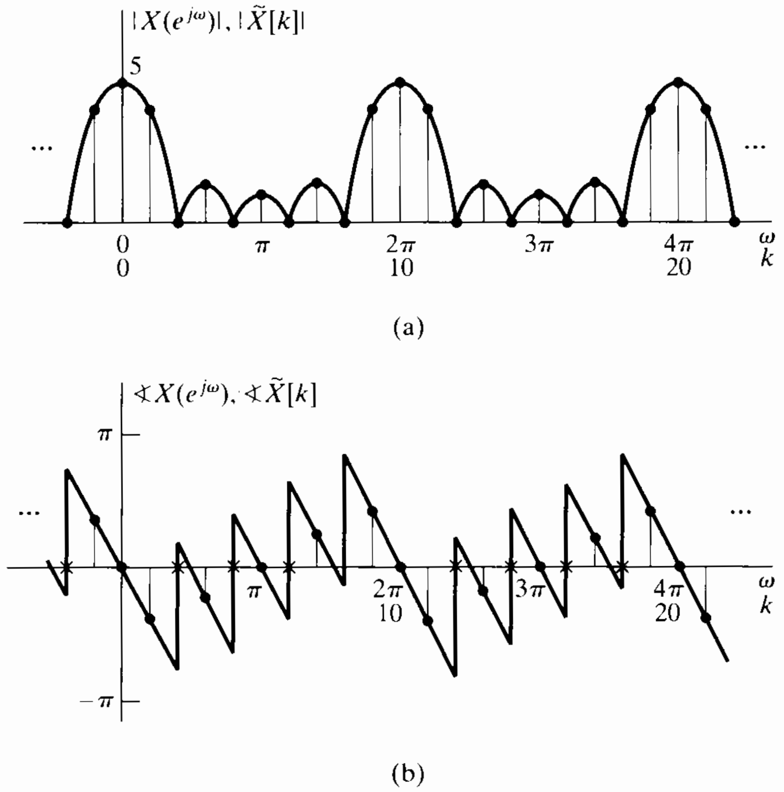

Figure 8.6 Overlay of Figures 8.2 and 8.5 illustrating the DFS coefficients of a periodic sequence as samples of the Fourier transform of one period.

8.4 SAMPLING THE FOURIER TRANSFORM

In this section, we discuss with more generality the relationship between an aperiodic sequence with Fourier transform $X(e^{j\omega})$ and the periodic sequence for which the DFS coefficients correspond to samples of $X(e^{j\omega})$ equally spaced in frequency. We will find this relationship to be particularly important when we discuss the discrete Fourier transform and its properties later in the chapter.

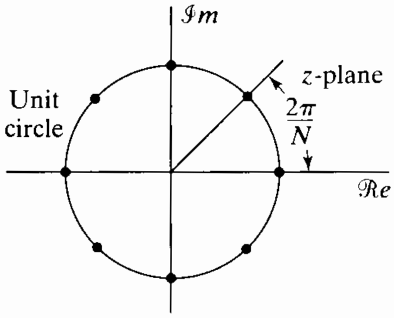

Figure 8.7 Points on the unit circle at which $X(z)$ is sampled to obtain the periodic sequence $\tilde{X}[k]$ ($N = 8$).

Consider an aperiodic sequence $x[n]$ with Fourier transform $X(e^{j\omega})$, and assume that a sequence $\tilde{X}[k]$ is obtained by sampling $X(e^{j\omega})$ at frequencies $\omega_k = 2\pi k/N$; i.e.,

$$\tilde{X}[k] = X(e^{j\omega})|_{\omega=(2\pi/N)k} = X(e^{j(2\pi/N)k}). \tag{8.49}$$

Since the Fourier transform is periodic in ω with period 2π, the resulting sequence is periodic in k with period N. Also, since the Fourier transform is equal to the z-transform evaluated on the unit circle, it follows that $\tilde{X}[k]$ can also be obtained by sampling $X(z)$ at N equally spaced points on the unit circle. Thus,

$$\tilde{X}[k] = X(z)|_{z=e^{j(2\pi/N)k}} = X(e^{j(2\pi/N)k}). \tag{8.50}$$

These sampling points are depicted in Figure 8.7 for $N = 8$. The figure makes it clear that the sequence of samples is periodic, since the N points are equally spaced starting with zero angle. Therefore, the same sequence repeats as k varies outside the range $0 \le k \le N-1$.

Note that the sequence of samples $\tilde{X}[k]$, being periodic with period N, *could* be the sequence of discrete Fourier series coefficients of a sequence $\tilde{x}[n]$. To obtain that sequence, we can simply substitute $\tilde{X}[k]$ obtained by sampling into Eq. (8.12):

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}[k] W_N^{-kn}. \tag{8.51}$$

Since we have made no assumption about $x[n]$ other than that the Fourier transform exists,

$$X(e^{j\omega}) = \sum_{m=-\infty}^{\infty} x[m]e^{-j\omega m}. \tag{8.52}$$

Substituting Eq. (8.52) into Eq. (8.49) and then substituting the resulting expression for $\tilde{X}[k]$ into Eq. (8.51) gives

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1}\left[\sum_{m=-\infty}^{\infty} x[m]e^{-j(2\pi/N)km}\right] W_N^{-kn}, \tag{8.53}$$

which, after we interchange the order of summation, becomes

$$\tilde{x}[n] = \sum_{m=-\infty}^{\infty} x[m]\left[\frac{1}{N}\sum_{k=0}^{N-1} W_N^{-k(n-m)}\right] = \sum_{m=-\infty}^{\infty} x[m]\tilde{p}[n-m]. \tag{8.54}$$

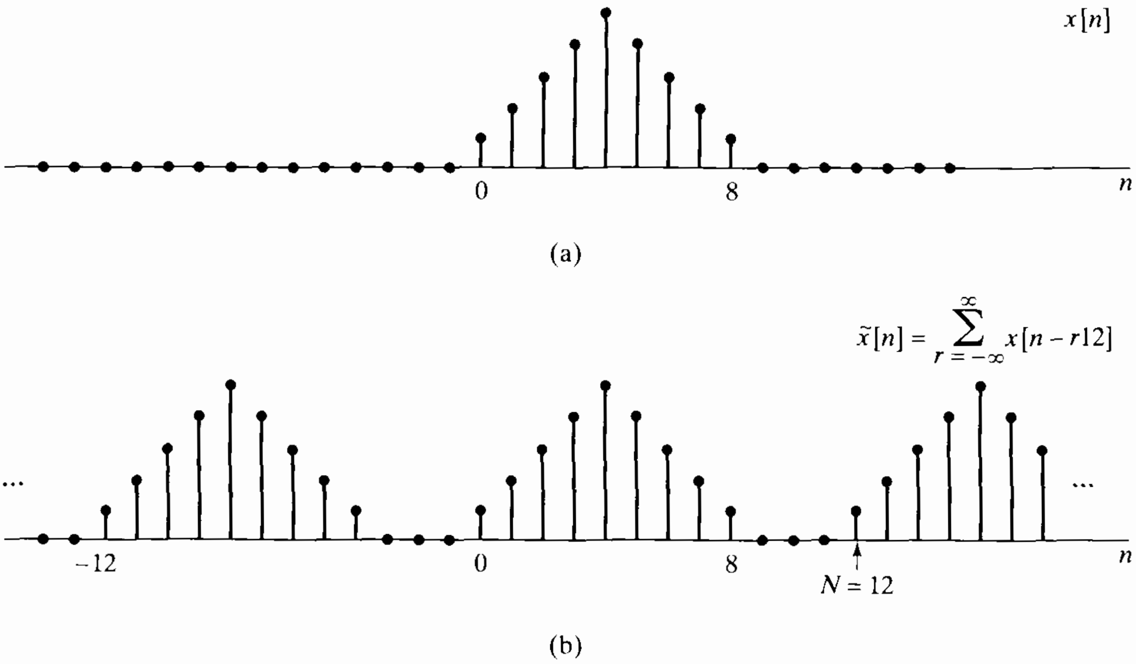

Figure 8.8 (a) Finite-length sequence $x[n]$. (b) Periodic sequence $\tilde{x}[n]$ corresponding to sampling the Fourier transform of $x[n]$ with $N = 12$.

The term in brackets in Eq. (8.54) can be seen from either Eq. (8.7) or Eq. (8.16) to be the Fourier series representation of the periodic impulse train of Examples 8.1 and 8.2. Specifically,

$$\tilde{p}[n-m] = \frac{1}{N}\sum_{k=0}^{N-1} W_N^{-k(n-m)} = \sum_{r=-\infty}^{\infty} \delta[n-m-rN] \tag{8.55}$$

and therefore,

$$\tilde{x}[n] = x[n] * \sum_{r=-\infty}^{\infty} \delta[n-rN] = \sum_{r=-\infty}^{\infty} x[n-rN], \tag{8.56}$$

where $*$ denotes aperiodic convolution. That is, $\tilde{x}[n]$ is the periodic sequence that results from the aperiodic convolution of $x[n]$ with a periodic unit-impulse train. Thus, the periodic sequence $\tilde{x}[n]$, corresponding to $\tilde{X}[k]$ obtained by sampling $X(e^{j\omega})$, is formed from $x[n]$ by adding together an infinite number of shifted replicas of $x[n]$. The shifts are all the positive and negative integer multiples of N, the period of the sequence $\tilde{X}[k]$. This is illustrated in Figure 8.8, where the sequence $x[n]$ is of length 9 and the value of N in Eq. (8.56) is $N = 12$. Consequently, the delayed replications of $x[n]$ do not overlap, and one period of the periodic sequence $\tilde{x}[n]$ is recognizable as $x[n]$. This is consistent with the discussion in Section 8.3 and Example 8.6 wherein we showed that the Fourier series coefficients for a periodic sequence are samples of the Fourier transform of one period. In Figure 8.9 the same sequence $x[n]$ is used, but the value of N is now $N = 7$. In this case the replicas of $x[n]$ overlap and one period of $\tilde{x}[n]$ is no longer identical to $x[n]$. In both cases, however, Eq. (8.49) still holds; i.e., in both

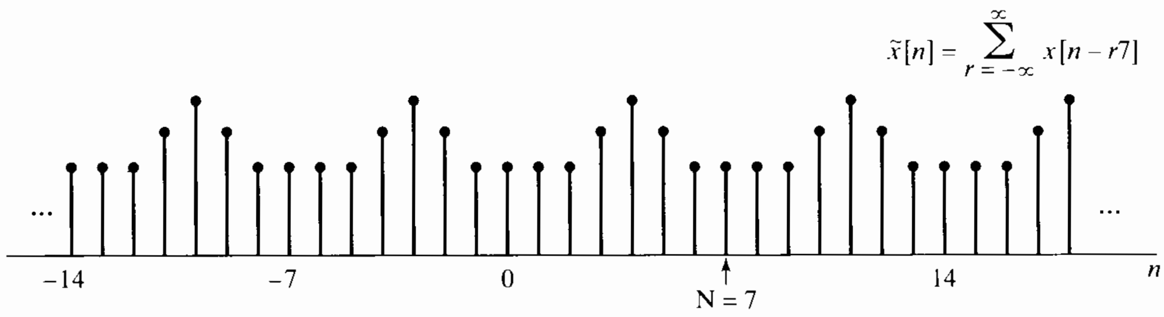

Figure 8.9 Periodic sequence $\tilde{x}[n]$ corresponding to sampling the Fourier transform of $x[n]$ in Figure 8.8(a) with $N = 7$.

cases the DFS coefficients of $\tilde{x}[n]$ are samples of the Fourier transform of $x[n]$ spaced in frequency at integer multiples of $2\pi/N$. This discussion should be reminiscent of our discussion of sampling in Chapter 4. The difference is that here we are sampling in the frequency domain rather than in the time domain. However, the general outlines of the mathematical representations are very similar.

For the example in Figure 8.8, the original sequence $x[n]$ can be recovered from $\tilde{x}[n]$ by extracting one period. Equivalently, the Fourier transform $X(e^{j\omega})$ can be recovered from the samples spaced in frequency by $2\pi/12$. In contrast, in Figure 8.9, $x[n]$ cannot be recovered by extracting one period of $\tilde{x}[n]$, and, equivalently, $X(e^{j\omega})$ cannot be recovered from its samples if the sample spacing is only $2\pi/7$. In effect, for the case illustrated in Figure 8.8, the Fourier transform of $x[n]$ has been sampled at a sufficiently small spacing (in frequency) to be able to recover it from these samples, whereas Figure 8.9 represents a case for which the Fourier transform has been undersampled. The relationship between $x[n]$ and one period of $\tilde{x}[n]$ in the undersampled case can be thought of as a form of aliasing in the time domain, essentially identical to the frequency-domain aliasing (discussed in Chapter 4) that results from undersampling in the time domain. Obviously, time-domain aliasing can be avoided only if $x[n]$ has finite length, just as frequency-domain aliasing can be avoided only for signals that have bandlimited Fourier transforms.

This discussion highlights several important concepts that will play a central role in the remainder of the chapter. We have seen that samples of the Fourier transform of an aperiodic sequence $x[n]$ can be thought of as DFS coefficients of a periodic sequence $\tilde{x}[n]$ obtained through summing periodic replicas of $x[n]$. If $x[n]$ has finite length and we take a sufficient number of equally spaced samples of its Fourier transform (specifically, a number greater than or equal to the length of $x[n]$), then the Fourier transform is recoverable from these samples, and, equivalently, $x[n]$ is recoverable from the corresponding periodic sequence $\tilde{x}[n]$ through the relation

$$x[n] = \begin{cases} \tilde{x}[n], & 0 \le n \le N-1, \\ 0, & \text{otherwise}. \end{cases} \tag{8.57}$$

A direct relationship between $X(e^{j\omega})$ and its samples $\tilde{X}[k]$, i.e., an interpolation formula for $X(e^{j\omega})$, can be derived (see Problem 8.54). However, the essence of our previous discussion is that to represent or to recover $x[n]$ it is not necessary to know

$X(e^{j\omega})$ at all frequencies if $x[n]$ has finite length. Given a finite-length sequence $x[n]$, we can form a periodic sequence using Eq. (8.56), which in turn can be represented by a discrete Fourier series. Alternatively, given the sequence of Fourier coefficients $\tilde{X}[k]$, we can find $\tilde{x}[n]$ and then use Eq. (8.57) to obtain $x[n]$. When the Fourier series is used in this way to represent finite-length sequences, it is called the discrete Fourier transform (DFT). In developing, discussing, and applying the DFT, it is always important to remember that the representation through samples of the Fourier transform is in effect a representation of the finite-duration sequence by a periodic sequence, one period of which is the finite-duration sequence that we wish to represent.

8.5 FOURIER REPRESENTATION OF FINITE-DURATION SEQUENCES: THE DISCRETE FOURIER TRANSFORM

In this section, we formalize the point of view suggested at the end of the previous section. We begin by considering a finite-length sequence $x[n]$ of length N samples such that $x[n] = 0$ outside the range $0 \le n \le N-1$. In many instances, we will want to assume that a sequence has length N even if its length is $M \le N$. In such cases, we simply recognize that the last $(N-M)$ samples are zero. To each finite-length sequence of length N, we can always associate a periodic sequence

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n-rN]. \tag{8.58a}$$

The finite-length sequence $x[n]$ can be recovered from $\tilde{x}[n]$ through Eq. (8.57), i.e.,

$$x[n] = \begin{cases} \tilde{x}[n], & 0 \le n \le N-1, \\ 0, & \text{otherwise.} \end{cases} \tag{8.58b}$$

Recall from Section 8.4 that the DFS coefficients of $\tilde{x}[n]$ are samples (spaced in frequency by $2\pi/N$) of the Fourier transform of $x[n]$. Since $x[n]$ is assumed to have finite length N, there is no overlap between the terms $x[n-rN]$ for different values of r. Thus, Eq. (8.58a) can alternatively be written as

$$\tilde{x}[n] = x[(n \text{ modulo } N)]. \tag{8.59}$$

For convenience, we will use the notation $((n))_N$ to denote (n modulo N); with this notation, Eq. (8.59) is expressed as

$$\tilde{x}[n] = x[((n))_N]. \tag{8.60}$$

Note that Eq. (8.60) is equivalent to Eq. (8.58a) *only* when $x[n]$ has length less than or equal to N. The finite-duration sequence $x[n]$ is obtained from $\tilde{x}[n]$ by extracting one period, as in Eq. (8.58b).

One informal and useful way of visualizing Eq. (8.59) is to think of wrapping a plot of the finite-duration sequence $x[n]$ around a cylinder with a circumference equal to the length of the sequence. As we repeatedly traverse the circumference of the cylinder, we see the finite-length sequence periodically repeated. With this interpretation, representation of the finite-length sequence by a periodic sequence corresponds to wrapping the sequence around the cylinder; recovering the finite-length sequence from the periodic sequence using Eq. (8.58b) can be visualized as unwrapping the cylinder and laying it flat so that the sequence is displayed on a linear time axis rather than a circular (modulo N) time axis.

As defined in Section 8.1, the sequence of discrete Fourier series coefficients $\tilde{X}[k]$ of the periodic sequence $\tilde{x}[n]$ is itself a periodic sequence with period N. To maintain a duality between the time and frequency domains, we will choose the Fourier coefficients that we associate with a finite-duration sequence to be a finite-duration sequence corresponding to one period of $\tilde{X}[k]$. This finite-duration sequence, $X[k]$, will be referred to as the discrete Fourier transform (DFT). Thus, the DFT, $X[k]$, is related to the DFS coefficients, $\tilde{X}[k]$, by

$$X[k] = \begin{cases} \tilde{X}[k], & 0 \le k \le N-1, \\ 0, & \text{otherwise}, \end{cases} \tag{8.61}$$

and

$$\tilde{X}[k] = X[(k \text{ modulo } N)] = X[((k))_N]. \tag{8.62}$$

From Section 8.1, $\tilde{X}[k]$ and $\tilde{x}[n]$ are related by

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{kn}, \tag{8.63}$$

$$\tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] W_N^{-kn}. \tag{8.64}$$

Since the summations in Eqs. (8.63) and (8.64) involve only the interval between zero and $(N-1)$, it follows from Eqs. (8.58b)–(8.64) that

$$X[k] = \begin{cases} \displaystyle\sum_{n=0}^{N-1} x[n] W_N^{kn}, & 0 \le k \le N-1, \\ 0, & \text{otherwise}, \end{cases} \tag{8.65}$$

$$x[n] = \begin{cases} \displaystyle\frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn}, & 0 \le n \le N-1, \\ 0, & \text{otherwise}. \end{cases} \tag{8.66}$$

Generally, the DFT analysis and synthesis equations are written as follows:

$$\text{Analysis equation:} \quad X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}, \tag{8.67}$$

$$\text{Synthesis equation:} \quad x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn}. \tag{8.68}$$

That is, the fact that $X[k] = 0$ for k outside the interval $0 \le k \le N-1$ and that $x[n] = 0$ for n outside the interval $0 \le n \le N-1$ is implied, but not always stated explicitly. The relationship between $x[n]$ and $X[k]$ implied by Eqs. (8.67) and (8.68) will sometimes be denoted as

$$x[n] \overset{\mathcal{DFT}}{\longleftrightarrow} X[k]. \tag{8.69}$$

In recasting Eqs. (8.11) and (8.12) in the form of Eqs. (8.67) and (8.68) for finite-duration sequences, we have not eliminated the inherent periodicity. As with the DFS, the DFT $X[k]$ is equal to samples of the periodic Fourier transform $X(e^{j\omega})$, and if Eq. (8.68) is evaluated for values of n outside the interval $0 \le n \le N-1$, the result will not be zero, but rather a periodic extension of $x[n]$. The inherent periodicity is always present. Sometimes it causes us difficulty and sometimes we can exploit it, but to totally ignore it is to invite trouble. In defining the DFT representation, we are simply recognizing that we are *interested* in values of $x[n]$ only in the interval $0 \le n \le N-1$ because $x[n]$ is really zero outside that interval, and we are *interested* in values of $X[k]$ only in the interval $0 \le k \le N-1$ because these are the only values needed in Eq. (8.68).

Example 8.7 The DFT of a Rectangular Pulse

To illustrate the DFT of a finite-duration sequence, consider $x[n]$ shown in Figure 8.10(a). In determining the DFT, we can consider $x[n]$ as a finite-duration sequence with any length greater than or equal to $N = 5$. Considered as a sequence of length $N = 5$, the periodic sequence $\tilde{x}[n]$ whose DFS corresponds to the DFT of $x[n]$ is shown in Figure 8.10(b). Since the sequence in Figure 8.10(b) is constant over the interval $0 \le n \le 4$, it follows that

$$\begin{aligned} \tilde{X}[k] &= \sum_{n=0}^{4} e^{-j(2\pi k/5)n} = \frac{1 - e^{-j2\pi k}}{1 - e^{-j(2\pi k/5)}} \\ &= \begin{cases} 5, & k = 0, \pm 5, \pm 10, \ldots, \\ 0, & \text{otherwise;} \end{cases} \end{aligned} \tag{8.70}$$

i.e., the only nonzero DFS coefficients $\tilde{X}[k]$ are at $k = 0$ and integer multiples of $k = 5$ (all of which represent the same complex exponential frequency). The DFS coefficients are shown in Figure 8.10(c). Also shown is the magnitude of the Fourier transform, $|X(e^{j\omega})|$. Clearly, $\tilde{X}[k]$ is a sequence of samples of $X(e^{j\omega})$ at frequencies

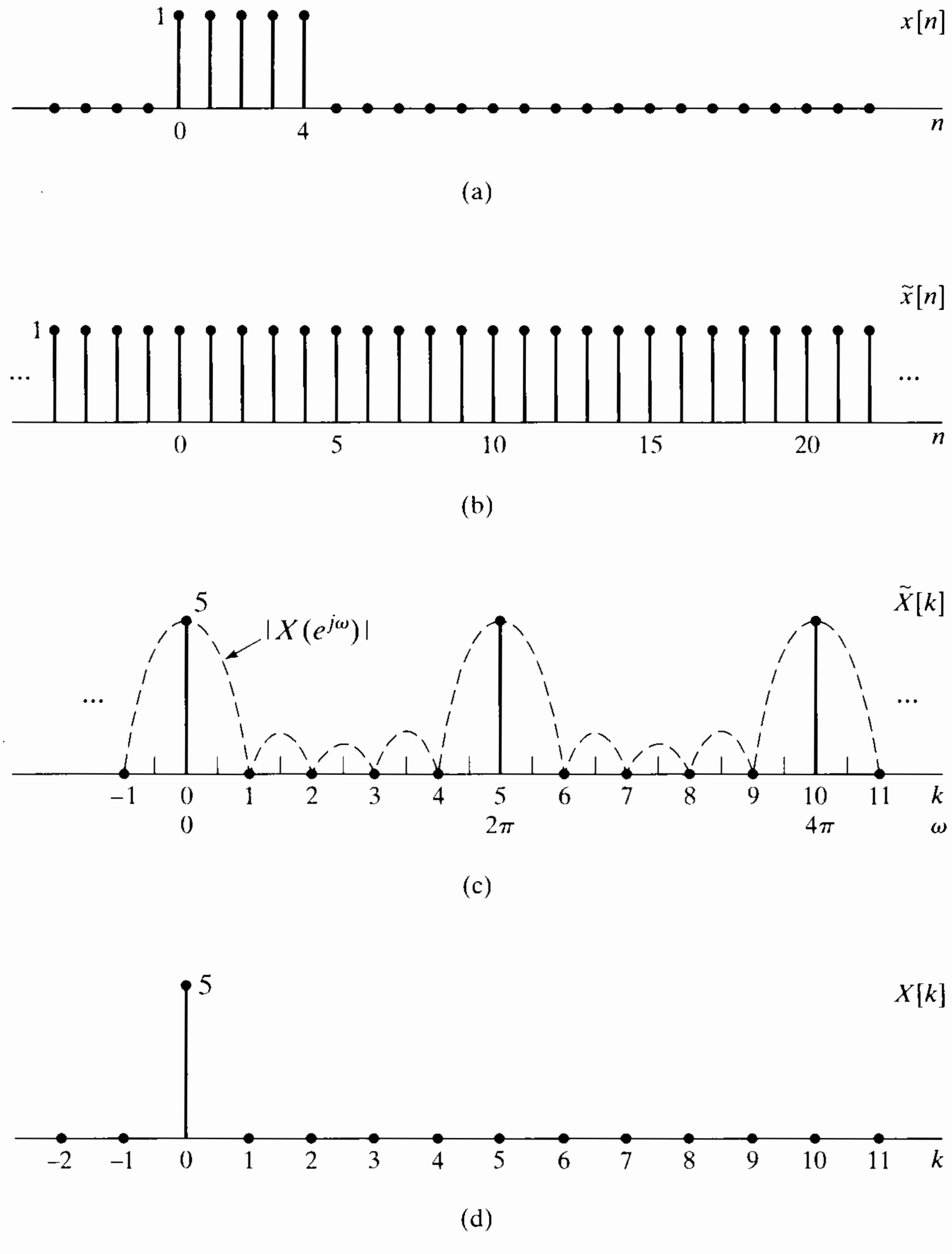

Figure 8.10 Illustration of the DFT. (a) Finite-length sequence $x[n]$. (b) Periodic sequence $\tilde{x}[n]$ formed from $x[n]$ with period $N = 5$. (c) Fourier series coefficients $\tilde{X}[k]$ for $\tilde{x}[n]$. To emphasize that the Fourier series coefficients are samples of the Fourier transform, $|X(e^{j\omega})|$ is also shown. (d) DFT of $x[n]$.

$\omega_k = 2\pi k/5$. According to Eq. (8.61), the five-point DFT of $x[n]$ corresponds to the finite-length sequence obtained by extracting one period of $\tilde{X}[k]$. Consequently, the five-point DFT of $x[n]$ is shown in Figure 8.10(d).

If, instead, we consider $x[n]$ to be of length $N = 10$, then the underlying periodic sequence is that shown in Figure 8.11(b), which is the periodic sequence considered

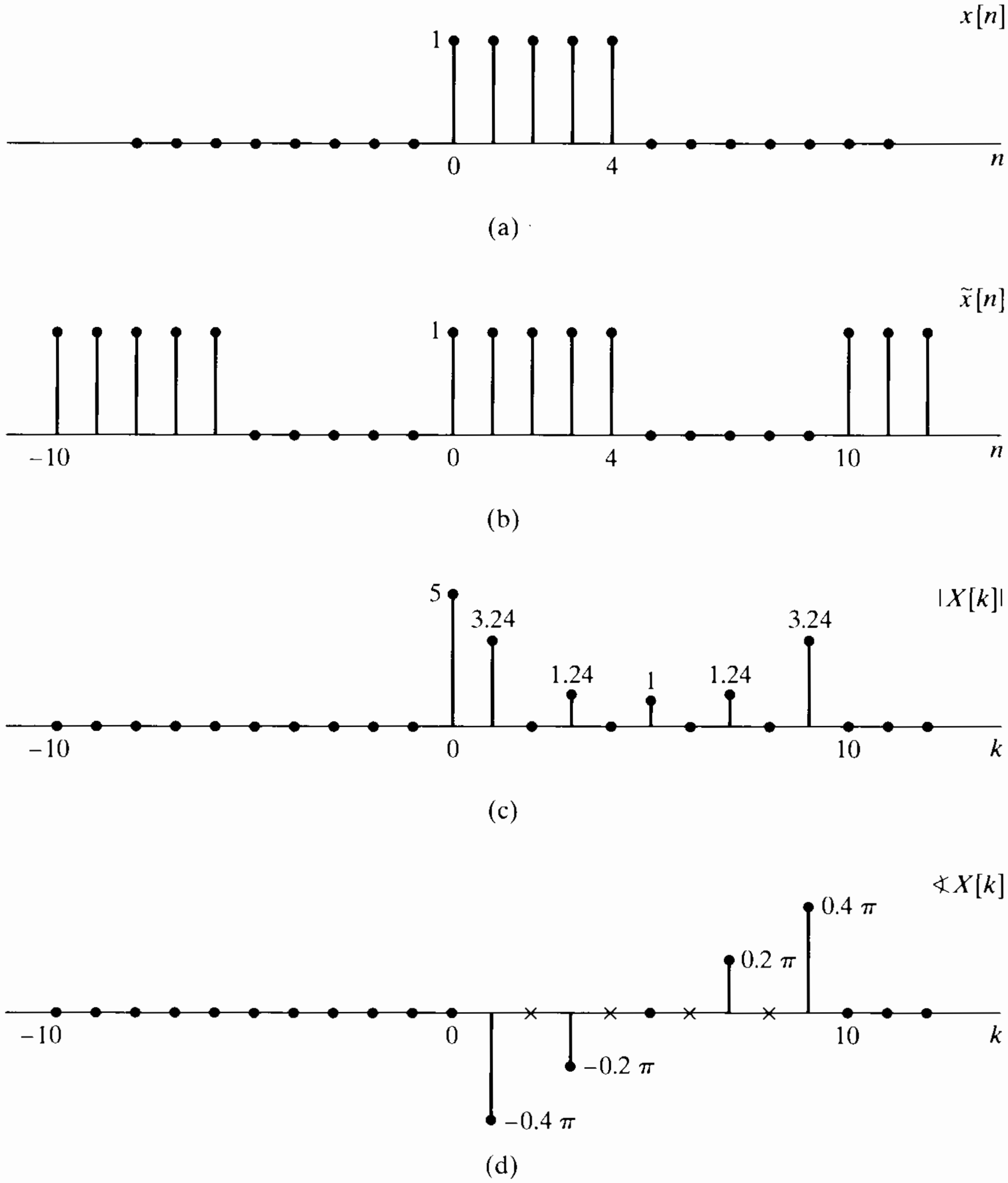

Figure 8.11 Illustration of the DFT. (a) Finite-length sequence $x[n]$. (b) Periodic sequence $\tilde{x}[n]$ formed from $x[n]$ with period $N = 10$. (c) DFT magnitude. (d) DFT phase. (x's indicate indeterminate values.)

in Example 8.3. Therefore, $\tilde{X}[k]$ is as shown in Figures 8.2 and 8.6, and the 10-point DFT $X[k]$ shown in Figures 8.11(c) and 8.11(d) is one period of $\tilde{X}[k]$.

The distinction between the finite-duration sequence $x[n]$ and the periodic sequence $\tilde{x}[n]$ related through Eqs. (8.57) and (8.60) may seem minor, since, by using these equations, it is straightforward to construct one from the other. However, the distinction becomes important in considering properties of the DFT and in considering the effect on $x[n]$ of modifications to $X[k]$. This will become evident in the next section, where we discuss the properties of the DFT representation.

8.6 PROPERTIES OF THE DISCRETE FOURIER TRANSFORM

In this section, we consider a number of properties of the DFT for finite-duration sequences. Our discussion parallels the discussion of Section 8.2 for periodic sequences. However, particular attention is paid to the interaction of the finite-length assumption and the implicit periodicity of the DFT representation of finite-length sequences.

8.6.1 Linearity

If two finite-duration sequences $x_1[n]$ and $x_2[n]$ are linearly combined, i.e., if

$$x_3[n] = ax_1[n] + bx_2[n], \tag{8.71}$$

then the DFT of $x_3[n]$ is

$$X_3[k] = aX_1[k] + bX_2[k]. \tag{8.72}$$

Clearly, if $x_1[n]$ has length N_1 and $x_2[n]$ has length N_2, then the maximum length of $x_3[n]$ will be $N_3 = \max[N_1, N_2]$. Thus, in order for Eq. (8.72) to be meaningful, both DFTs must be computed with the same length $N \geq N_3$. If, for example, $N_1 < N_2$, then $X_1[k]$ is the DFT of the sequence $x_1[n]$ augmented by $(N_2 - N_1)$ zeros. That is, the N_2-point DFT of $x_1[n]$ is

$$X_1[k] = \sum_{n=0}^{N_1-1} x_1[n]W_{N_2}^{kn}, \qquad 0 \leq k \leq N_2 - 1, \tag{8.73}$$

and the N_2-point DFT of $x_2[n]$ is

$$X_2[k] = \sum_{n=0}^{N_2-1} x_2[n]W_{N_2}^{kn}, \qquad 0 \leq k \leq N_2 - 1. \tag{8.74}$$

In summary, if

$$x_1[n] \overset{\mathcal{DFT}}{\longleftrightarrow} X_1[k] \tag{8.75a}$$

and

$$x_2[n] \overset{\mathcal{DFT}}{\longleftrightarrow} X_2[k], \tag{8.75b}$$

then

$$ax_1[n] + bx_2[n] \overset{\mathcal{DFT}}{\longleftrightarrow} aX_1[k] + bX_2[k], \tag{8.76}$$

where the lengths of the sequences and their discrete Fourier transforms are all equal to the maximum of the lengths of $x_1[n]$ and $x_2[n]$. Of course, DFTs of greater length can be computed by augmenting *both* sequences with zero-valued samples.

8.6.2 Circular Shift of a Sequence

According to Section 2.9.2 and property 2 in Table 2.2, if $X(e^{j\omega})$ is the Fourier transform of $x[n]$, then $e^{-j\omega m}X(e^{j\omega})$ is the Fourier transform of the time-shifted sequence $x[n-m]$. In other words, a shift in the time domain by m points (with positive m corresponding to a time delay and negative m to a time advance) corresponds in the frequency domain to multiplication of the Fourier transform by the linear phase factor $e^{-j\omega m}$. In Section 8.2.2, we discussed the corresponding property for the DFS coefficients of a periodic sequence; specifically, if a periodic sequence $\tilde{x}[n]$ has Fourier series coefficients $\tilde{X}[k]$, then the shifted sequence $\tilde{x}[n-m]$ has Fourier series coefficients $e^{-j(2\pi k/N)m}\tilde{X}[k]$. Now we will consider the operation in the time domain that corresponds to multiplying the DFT coefficients of a finite-length sequence $x[n]$ by the linear phase factor $e^{-j(2\pi k/N)m}$. Specifically, let $x_1[n]$ denote the finite-length sequence for which the DFT is $e^{-j(2\pi k/N)m}X[k]$; i.e., if

$$x[n] \overset{\mathcal{DFT}}{\longleftrightarrow} X[k], \tag{8.77}$$

then we are interested in $x_1[n]$ such that

$$x_1[n] \overset{\mathcal{DFT}}{\longleftrightarrow} X_1[k] = e^{-j(2\pi k/N)m}X[k]. \tag{8.78}$$

Since the N-point DFT represents a finite-duration sequence of length N, both $x[n]$ and $x_1[n]$ must be zero outside the interval $0 \le n \le N-1$, and consequently, $x_1[n]$ cannot result from a simple time shift of $x[n]$. The correct result follows directly from the result of Section 8.2.2 and the interpretation of the DFT as the Fourier series coefficients of the periodic sequence $x_1[((n))_N]$. In particular, from Eqs. (8.59) and (8.62) it follows that

$$\tilde{x}[n] = x[((n))_N] \overset{\mathcal{DFS}}{\longleftrightarrow} \tilde{X}[k] = X[((k))_N], \tag{8.79}$$

and similarly, we can define a periodic sequence $\tilde{x}_1[n]$ such that

$$\tilde{x}_1[n] = x_1[((n))_N] \overset{\mathcal{DFS}}{\longleftrightarrow} \tilde{X}_1[k] = X_1[((k))_N], \tag{8.80}$$

where, by assumption,

$$X_1[k] = e^{-j(2\pi k/N)m}X[k]. \tag{8.81}$$

Therefore, the discrete Fourier series coefficients of $\tilde{x}_1[n]$ are

$$\tilde{X}_1[k] = e^{-j[2\pi((k))_N/N]m}X[((k))_N]. \tag{8.82}$$

Note that

$$e^{-j[2\pi((k))_N/N]m} = e^{-j(2\pi k/N)m}. \tag{8.83}$$

That is, since $e^{-j(2\pi k/N)m}$ is periodic with period N in both k and m, we can drop the notation $((k))_N$. Hence, Eq. (8.82) becomes

$$\tilde{X}_1[k] = e^{-j(2\pi k/N)m}\tilde{X}[k], \tag{8.84}$$

so that it follows from Section 8.2.2 that

$$\tilde{x}_1[n] = \tilde{x}[n-m] = x[((n-m))_N]. \tag{8.85}$$

Thus, the finite-length sequence $x_1[n]$ whose DFT is given by Eq. (8.81) is

$$x_1[n] = \begin{cases} \tilde{x}_1[n] = x[((n-m))_N], & 0 \le n \le N-1, \\ 0, & \text{otherwise.} \end{cases} \tag{8.86}$$

Equation (8.86) tells us how to construct $x_1[n]$.

Example 8.8 Circular Shift of a Sequence

The circular shift procedure is illustrated in Figure 8.12 for $m = -2$; i.e., we want to determine $x_1[n] = x[((n+2))_N]$ for $N = 6$, which we have shown will have DFT $X_1[k] = W_6^{-2k}X[k]$. Specifically, from $x[n]$, we construct the periodic sequence $\tilde{x}[n] = x[((n))_6]$, as indicated in Figure 8.12(b). According to Eq. (8.85), we then shift $\tilde{x}[n]$ by 2 to the left, obtaining $\tilde{x}_1[n] = \tilde{x}[n+2]$ as in Figure 8.12(c). Finally, using Eq. (8.86), we extract one period of $\tilde{x}_1[n]$ to obtain $x_1[n]$, as indicated in Figure 8.12(d).

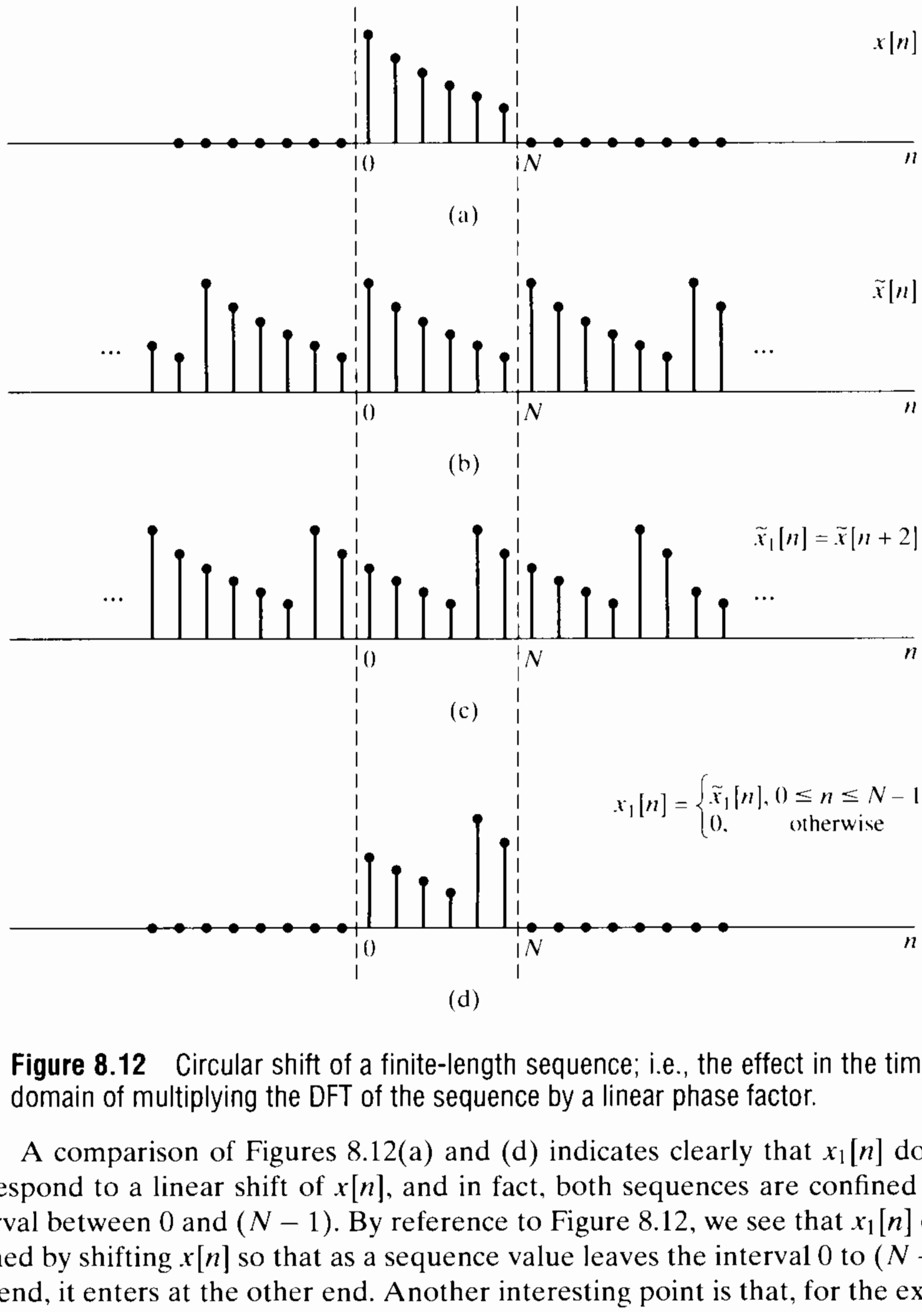

Figure 8.12 Circular shift of a finite-length sequence; i.e., the effect in the time domain of multiplying the DFT of the sequence by a linear phase factor.

A comparison of Figures 8.12(a) and (d) indicates clearly that $x_1[n]$ does *not* correspond to a linear shift of $x[n]$, and in fact, both sequences are confined to the interval between 0 and $(N-1)$. By reference to Figure 8.12, we see that $x_1[n]$ can be formed by shifting $x[n]$ so that as a sequence value leaves the interval 0 to $(N-1)$ at one end, it enters at the other end. Another interesting point is that, for the example shown in Figure 8.12(a), if we form $x_2[n] = x[((n-4))_6]$ by shifting the sequence by 4 to the right modulo 6, we obtain the same sequence as $x_1[n]$. In terms of the DFT, this results because $W_6^{4k} = W_6^{-2k}$ or, more generally, $W_N^{mk} = W_N^{-(N-m)k}$, which implies that an N-point circular shift in one direction by m is the same as a circular shift in the opposite direction by $N - m$.

In Section 8.4, we suggested the interpretation of forming the periodic sequence $\tilde{x}[n]$ from the finite-length sequence $x[n]$ by displaying $x[n]$ around the circumference of a cylinder with a circumference of exactly N points. As we repeatedly traverse the circumference of the cylinder, the sequence that we see is the periodic sequence $\tilde{x}[n]$. A linear shift of this sequence corresponds, then, to a *rotation* of the cylinder. In the context of finite-length sequences and the DFT, such a shift is called a *circular* shift or a *rotation* of the sequence in the interval $0 \leq n \leq N-1$.

In summary, the circular shift property of the DFT is

$$x[((n-m))_N], \quad 0 \leq n \leq N-1 \overset{\mathcal{DFT}}{\longleftrightarrow} e^{-j(2\pi k/N)m}X[k]. \tag{8.87}$$

8.6.3 Duality

Since the DFT is so closely associated with the DFS, we would expect the DFT to exhibit a duality property similar to that of the DFS discussed in Section 8.2.3. In fact, from an examination of Eqs. (8.67) and (8.68), we see that the analysis and synthesis equations differ only in the factor $1/N$ and the sign of the exponent of the powers of W_N.

The DFT duality property can be derived by exploiting the relationship between the DFT and the DFS as in our derivation of the circular shift property. Toward this end, consider $x[n]$ and its DFT $X[k]$, and construct the periodic sequences

$$\tilde{x}[n] = x[((n))_N], \tag{8.88a}$$

$$\tilde{X}[k] = X[((k))_N], \tag{8.88b}$$

so that

$$\tilde{x}[n] \overset{\mathcal{DFS}}{\longleftrightarrow} \tilde{X}[k]. \tag{8.89}$$

From the duality property given in Eqs. (8.25),

$$\tilde{X}[n] \overset{\mathcal{DFS}}{\longleftrightarrow} N\tilde{x}[-k]. \tag{8.90}$$

If we define the periodic sequence $\tilde{x}_1[n] = \tilde{X}[n]$, one period of which is the finite-length sequence $x_1[n] = X[n]$, then the DFS coefficients of $\tilde{x}_1[n]$ are $\tilde{X}_1[k] = N\tilde{x}[-k]$. Therefore, the DFT of $x_1[n]$ is

$$X_1[k] = \begin{cases} N\tilde{x}[-k], & 0 \leq k \leq N-1, \\ 0, & \text{otherwise}, \end{cases} \tag{8.91}$$

or, equivalently,

$$X_1[k] = \begin{cases} Nx[((-k))_N], & 0 \leq k \leq N-1, \\ 0, & \text{otherwise}. \end{cases} \tag{8.92}$$

Consequently, the duality property for the DFT can be expressed as follows: If

$$x[n] \overset{\mathcal{DFT}}{\longleftrightarrow} X[k], \tag{8.93a}$$

then

$$X[n] \overset{\mathcal{DFT}}{\longleftrightarrow} Nx[((-k))_N], \quad 0 \leq k \leq N-1. \tag{8.93b}$$

The sequence $Nx[((-k))_N]$ is $Nx[k]$ index reversed, modulo N. As in the case of shifting modulo N, the process of index reversing modulo N is usually best visualized in terms of the underlying periodic sequences.

Example 8.9 The Duality Relationship for the DFT

To illustrate the duality relationship in Eqs. (8.93), let us consider the sequence $x[n]$ of Example 8.7. Figure 8.13(a) shows the finite-length sequence $x[n]$, and Figures 8.13(b) and 8.13(c) are the real and imaginary parts, respectively, of the corresponding 10-point DFT $X[k]$. By simply relabeling the horizontal axis, we obtain the complex sequence $x_1[n] = X[n]$, as shown in Figures 8.13(d) and 8.13(e). According to the duality relation in Eqs. (8.93), the 10-point DFT of the (complex-valued) sequence $X[n]$ is the sequence shown in Figure 8.13(f).

8.6.4 Symmetry Properties

Since the DFT of $x[n]$ is identical to the DFS coefficients of the periodic sequence $\tilde{x}[n] = x[((n))_N]$, symmetry properties associated with the DFT can be inferred from the symmetry properties of the DFS summarized in Table 8.1 in Section 8.2.6. Specifically, using Eqs. (8.88) together with properties 9 and 10 in Table 8.1, we have

$$x^*[n] \overset{\mathcal{DFT}}{\longleftrightarrow} X^*[((-k))_N], \qquad 0 \le n \le N-1, \tag{8.94}$$

and

$$x^*[((-n))_N] \overset{\mathcal{DFT}}{\longleftrightarrow} X^*[k], \qquad 0 \le n \le N-1. \tag{8.95}$$

Properties 11–14 in Table 8.1 refer to the decomposition of a periodic sequence into the sum of a conjugate-symmetric and a conjugate-antisymmetric sequence. This suggests the decomposition of the finite-duration sequence $x[n]$ into the two finite-duration sequences of duration N corresponding to one period of the conjugate-symmetric and one period of the conjugate-antisymmetric components of $\tilde{x}[n]$. We will denote these components of $x[n]$ as $x_{\text{ep}}[n]$ and $x_{\text{op}}[n]$. Thus, with

$$\tilde{x}[n] = x[((n))_N] \tag{8.96}$$

and the conjugate-symmetric part being

$$\tilde{x}_e[n] = \tfrac{1}{2}\{\tilde{x}[n] + \tilde{x}^*[-n]\}, \tag{8.97}$$

and the conjugate-antisymmetric part being

$$\tilde{x}_o[n] = \tfrac{1}{2}\{\tilde{x}[n] - \tilde{x}^*[-n]\}, \tag{8.98}$$

we define $x_{\text{ep}}[n]$ and $x_{\text{op}}[n]$ as

$$x_{\text{ep}}[n] = \tilde{x}_e[n], \qquad 0 \le n \le N-1, \tag{8.99}$$

$$x_{\text{op}}[n] = \tilde{x}_o[n], \qquad 0 \le n \le N-1, \tag{8.100}$$

or, equivalently,

$$x_{\text{ep}}[n] = \tfrac{1}{2}\{x[((n))_N] + x^*[((-n))_N]\}, \qquad 0 \le n \le N-1, \tag{8.101a}$$

$$x_{\text{op}}[n] = \tfrac{1}{2}\{x[((n))_N] - x^*[((-n))_N]\}, \qquad 0 \le n \le N-1, \tag{8.101b}$$

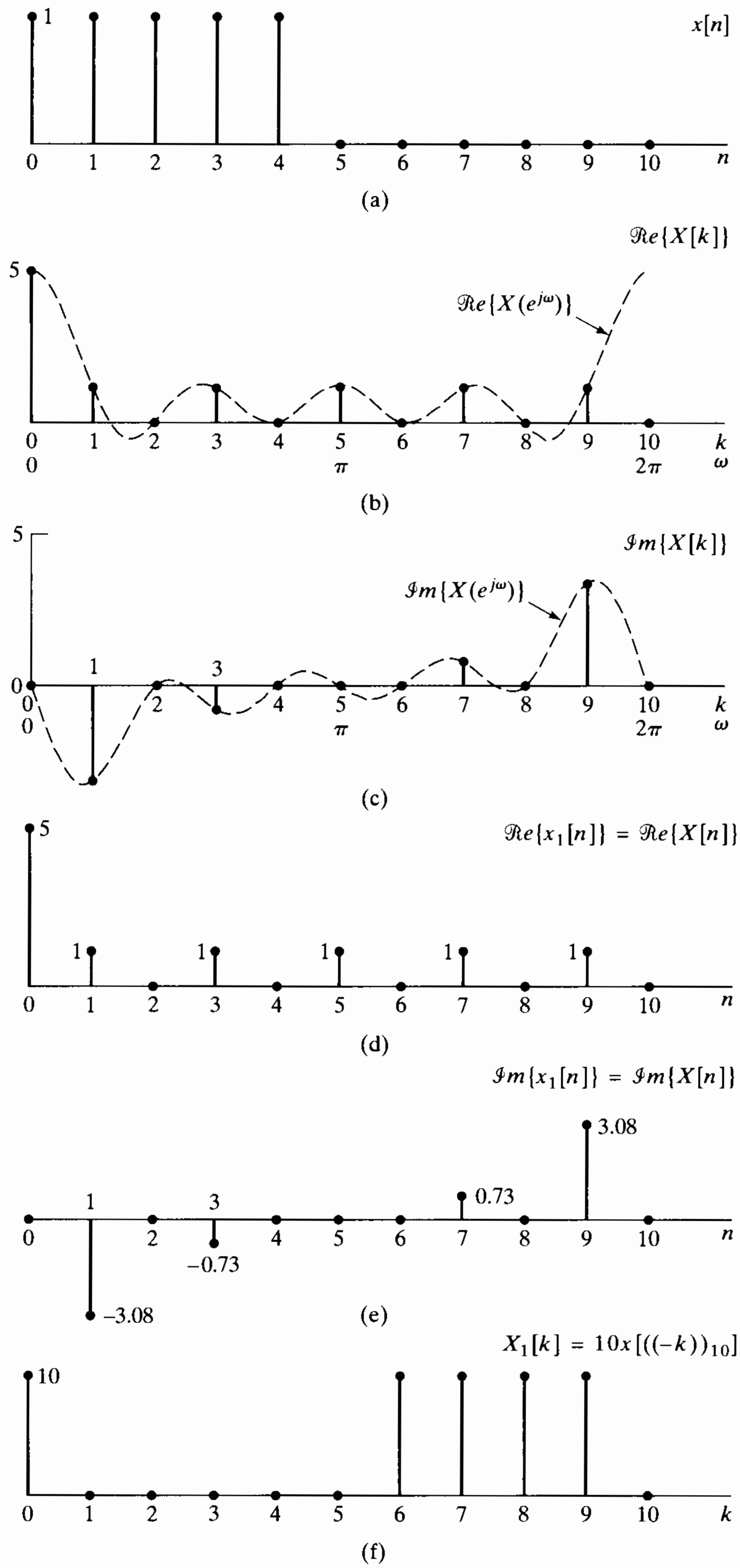

Figure 8.13 Illustration of duality. (a) Real finite-length sequence $x[n]$. (b) and (c) Real and imaginary parts of corresponding DFT $X[k]$. (d) and (e) The real and imaginary parts of the dual sequence $x_1[n] = X[n]$. (f) The DFT of $x_1[n]$.

with both $x_{\text{ep}}[n]$ and $x_{\text{op}}[n]$ being finite-length sequences, i.e., both zero outside the interval $0 \le n \le N-1$. Since $((-n))_N = (N-n)$ and $((n))_N = n$ for $0 \le n \le N-1$, we can also express Eqs. (8.101) as

$$x_{\text{ep}}[n] = \tfrac{1}{2}\{x[n] + x^*[N-n]\}, \qquad 1 \le n \le N-1, \tag{8.102a}$$

$$x_{\text{ep}}[0] = \mathcal{R}e\{x[0]\}, \tag{8.102b}$$

$$x_{\text{op}}[n] = \tfrac{1}{2}\{x[n] - x^*[N-n]\}, \qquad 1 \le n \le N-1, \tag{8.102c}$$

$$x_{\text{op}}[0] = j\mathcal{J}m\{x[0]\}. \tag{8.102d}$$

This form of the equations is convenient, since it avoids the modulo N computation of indices.

Clearly, $x_{\text{ep}}[n]$ and $x_{\text{op}}[n]$ are not equivalent to $x_e[n]$ and $x_o[n]$ as defined by Eq. (2.154). However, it can be shown (see Problem 8.56) that

$$x_{\text{ep}}[n] = \{x_e[n] + x_e[n-N]\}, \qquad 0 \le n \le N-1, \tag{8.103}$$

and

$$x_{\text{op}}[n] = \{x_o[n] + x_o[n-N]\}, \qquad 0 \le n \le N-1. \tag{8.104}$$

In other words, $x_{\text{ep}}[n]$ and $x_{\text{op}}[n]$ can be generated by aliasing $x_e[n]$ and $x_o[n]$ into the interval $0 \le n \le N-1$. The sequences $x_{\text{ep}}[n]$ and $x_{\text{op}}[n]$ will be referred to as *the periodic conjugate-symmetric* and *periodic conjugate-antisymmetric* components, respectively, of $x[n]$. When $x_{\text{ep}}[n]$ and $x_{\text{op}}[n]$ are real, they will be referred to as the *periodic even* and *periodic odd* components, respectively. Note that the sequences $x_{\text{ep}}[n]$ and $x_{\text{op}}[n]$ are *not* periodic sequences; they are, however, finite-length sequences that are equal to one period of the periodic sequences $\tilde{x}_e[n]$ and $\tilde{x}_o[n]$, respectively.

Equations (8.101) and (8.102) define $x_{\text{ep}}[n]$ and $x_{\text{op}}[n]$ in terms of $x[n]$. The inverse relation, expressing $x[n]$ in terms of $x_{\text{ep}}[n]$ and $x_{\text{op}}[n]$, can be obtained by using Eqs. (8.97) and (8.98) to express $\tilde{x}[n]$ as

$$\tilde{x}[n] = \tilde{x}_e[n] + \tilde{x}_o[n]. \tag{8.105}$$

Thus,

$$x[n] = \tilde{x}[n] = \tilde{x}_e[n] + \tilde{x}_o[n], \qquad 0 \le n \le N-1. \tag{8.106}$$

Combining Eqs. (8.106) with Eqs. (8.99) and (8.100), we obtain

$$x[n] = x_{\text{ep}}[n] + x_{\text{op}}[n]. \tag{8.107}$$

Alternatively, Eqs. (8.102), when added, also lead to Eq. (8.107). The symmetry properties of the DFT associated with properties 11–14 in Table 8.1 now follow in a straightforward way:

$$\mathcal{R}e\{x[n]\} \overset{\mathcal{DFT}}{\longleftrightarrow} X_{\text{ep}}[k], \tag{8.108}$$

$$j\mathcal{J}m\{x[n]\} \overset{\mathcal{DFT}}{\longleftrightarrow} X_{\text{op}}[k], \tag{8.109}$$

$$x_{\text{ep}}[n] \overset{\mathcal{DFT}}{\longleftrightarrow} \mathcal{R}e\{X[k]\}, \tag{8.110}$$

$$x_{\text{op}}[n] \overset{\mathcal{DFT}}{\longleftrightarrow} j\mathcal{J}m\{X[k]\}. \tag{8.111}$$

8.6.5 Circular Convolution

In Section 8.2.5, we showed that multiplication of the DFS coefficients of two periodic sequences corresponds to a periodic convolution of the sequences. Here we consider two *finite-duration* sequences $x_1[n]$ and $x_2[n]$, both of length N, with DFTs $X_1[k]$ and $X_2[k]$, respectively, and we wish to determine the sequence $x_3[n]$ for which the DFT is $X_3[k] = X_1[k]X_2[k]$. To determine $x_3[n]$, we can apply the results of Section 8.2.5. Specifically, $x_3[n]$ corresponds to one period of $\tilde{x}_3[n]$, which is given by Eq. (8.27). Thus,

$$x_3[n] = \sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n-m], \qquad 0 \le n \le N-1, \tag{8.112}$$

or, equivalently,

$$x_3[n] = \sum_{m=0}^{N-1} x_1[((m))_N]x_2[((n-m))_N], \qquad 0 \le n \le N-1. \tag{8.113}$$

Since $((m))_N = m$ for $0 \le m \le N-1$, Eq. (8.113) can be written

$$x_3[n] = \sum_{m=0}^{N-1} x_1[m]x_2[((n-m))_N], \qquad 0 \le n \le N-1. \tag{8.114}$$

Equations (8.112) and (8.114) differ from a linear convolution of $x_1[n]$ and $x_2[n]$ as defined by Eq. (2.52) in some important respects. In linear convolution, the computation of the sequence value $x_3[n]$ involves multiplying one sequence by a time-reversed and linearly shifted version of the other and then summing the values of the product $x_1[m]x_2[n-m]$ over all m. To obtain successive values of the sequence formed by the convolution operation, the two sequences are successively shifted relative to each other. In contrast, for the convolution defined by Eq. (8.114), the second sequence is circularly time reversed and circularly shifted with respect to the first. For this reason, the operation of combining two finite-length sequences according to Eq. (8.114) is called *circular convolution*. More specifically, we refer to Eq. (8.114) as an *N-point circular convolution*, explicity identifying the fact that both sequences have length N (or less) and that the sequences are shifted modulo N. Sometimes the operation of forming a sequence $x_3[n]$ for $0 \le n \le N-1$ using Eq. (8.114) will be denoted

$$x_3[n] = x_1[n] \circledN x_2[n]. \tag{8.115}$$

Since the DFT of $x_3[n]$ is $X_3[k] = X_1[k]X_2[k]$ and since $X_1[k]X_2[k] = X_2[k]X_1[k]$, it follows with no further analysis that

$$x_3[n] = x_2[n] \circledN x_1[n], \tag{8.116}$$

or, more specifically,

$$x_3[n] = \sum_{m=0}^{N-1} x_2[m]x_1[((n-m))_N]. \tag{8.117}$$

That is, circular convolution, like linear convolution, is a commutative operation.

Since circular convolution is really just periodic convolution, Example 8.4 and Figure 8.3 are also illustrative of circular convolution. However, if we utilize the notion of circular shifting, it is not necessary to construct the underlying periodic sequences as in Figure 8.3. This is illustrated in the following examples.

Example 8.10 Circular Convolution with a Delayed Impulse Sequence

An example of circular convolution is provided by the result of Section 8.6.2. Let $x_2[n]$ be a finite-duration sequence of length N and

$$x_1[n] = \delta[n - n_0], \tag{8.118}$$

where $0 < n_0 < N$. Clearly, $x_1[n]$ can be considered as the finite-duration sequence

$$x_1[n] = \begin{cases} 0, & 0 \le n < n_0, \\ 1, & n = n_0, \\ 0, & n_0 < n \le N - 1. \end{cases} \tag{8.119}$$

as depicted in Figure 8.14 for $n_0 = 1$.

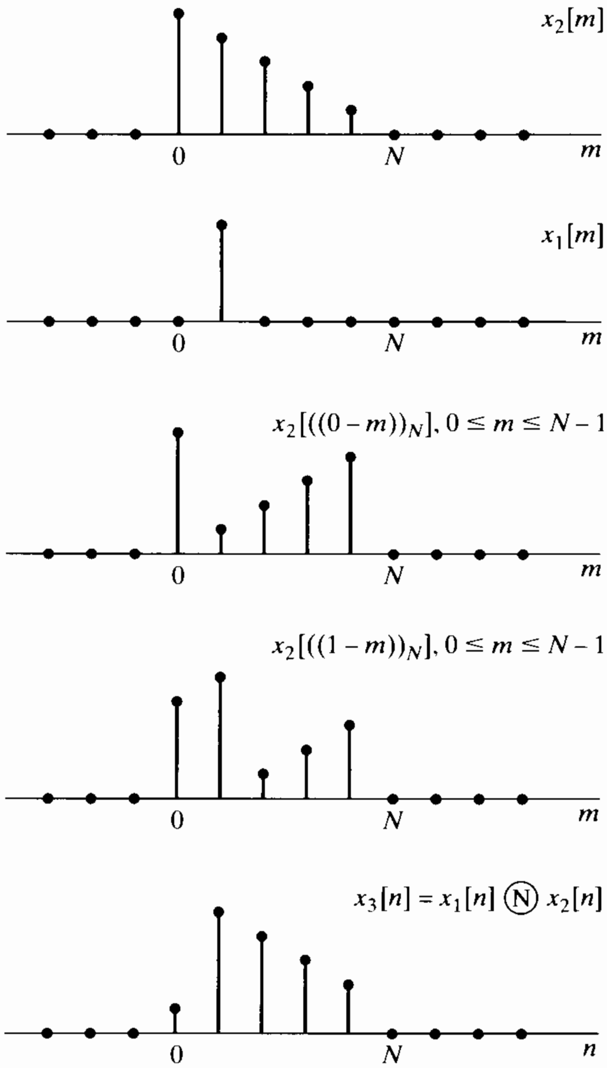

Figure 8.14 Circular convolution of a finite-length sequence $x_2[n]$ with a single delayed impulse, $x_1[n] = \delta[n - 1]$.

The DFT of $x_1[n]$ is

$$X_1[k] = W_N^{kn_0}. \tag{8.120}$$

If we form the product

$$X_3[k] = W_N^{kn_0} X_2[k], \tag{8.121}$$

we see from Section 8.6.2 that the finite-duration sequence corresponding to $X_3[k]$ is the sequence $x_2[n]$ rotated to the right by n_0 samples in the interval $0 \le n \le N-1$. That is, the circular convolution of a sequence $x_2[n]$ with a single delayed unit impulse results in a rotation of $x_2[n]$ in the interval $0 \le n \le N-1$. This example is illustrated in Figure 8.14 for $N = 5$ and $n_0 = 1$. Here we show the sequences $x_2[m]$ and $x_1[m]$ and then $x_2[((0-m))_N]$ and $x_2[((1-m))_N]$. It is clear from these two cases that the result of circular convolution of $x_2[n]$ with a single shifted unit impulse will be to circularly shift $x_2[n]$. The last sequence shown is $x_3[n]$, the result of the circular convolution of $x_1[n]$ and $x_2[n]$.

Example 8.11 Circular Convolution of Two Rectangular Pulses

As another example of circular convolution, let

$$x_1[n] = x_2[n] = \begin{cases} 1, & 0 \le n \le L-1, \\ 0, & \text{otherwise}, \end{cases} \tag{8.122}$$

where, in Figure 8.15, $L = 6$. If we let N denote the DFT length, then, for $N = L$, the N-point DFTs are

$$X_1[k] = X_2[k] = \sum_{n=0}^{N-1} W_N^{kn} = \begin{cases} N, & k = 0, \\ 0, & \text{otherwise}. \end{cases} \tag{8.123}$$

If we explicitly multiply $X_1[k]$ and $X_2[k]$, we obtain

$$X_3[k] = X_1[k]X_2[k] = \begin{cases} N^2, & k = 0, \\ 0, & \text{otherwise}, \end{cases} \tag{8.124}$$

from which it follows that

$$x_3[n] = N, \qquad 0 \le n \le N-1. \tag{8.125}$$

This result is depicted in Figure 8.15. Clearly, as the sequence $x_2[((n-m))_N]$ is rotated with respect to $x_1[m]$, the sum of products $x_1[m]x_2[((n-m))_N]$ will always be equal to N.

It is, of course, possible to consider $x_1[n]$ and $x_2[n]$ as $2L$-point sequences by augmenting them with L zeros. If we then perform a $2L$-point circular convolution of the augmented sequences, we obtain the sequence in Figure 8.16, which can be seen to be identical to the linear convolution of the finite-duration sequences $x_1[n]$ and $x_2[n]$. This important observation will be discussed in much more detail in Section 8.7.

Note that for $N = 2L$, as in Figure 8.16,

$$X_1[k] = X_2[k] = \frac{1 - W_N^{Lk}}{1 - W_N^k},$$

so the DFT of the triangular-shaped sequence $x_3[n]$ in Figure 8.16(e) is

$$X_3[k] = \left(\frac{1 - W_N^{Lk}}{1 - W_N^k}\right)^2,$$

with $N = 2L$.

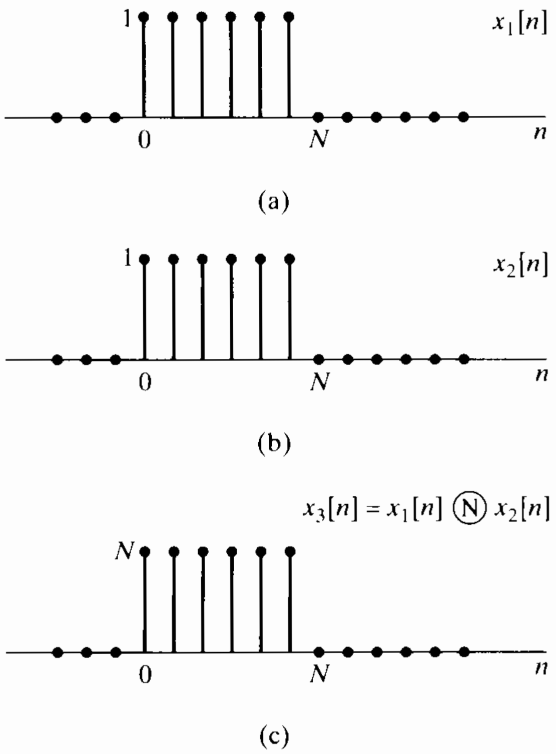

Figure 8.15 N-point circular convolution of two constant sequences of length N.

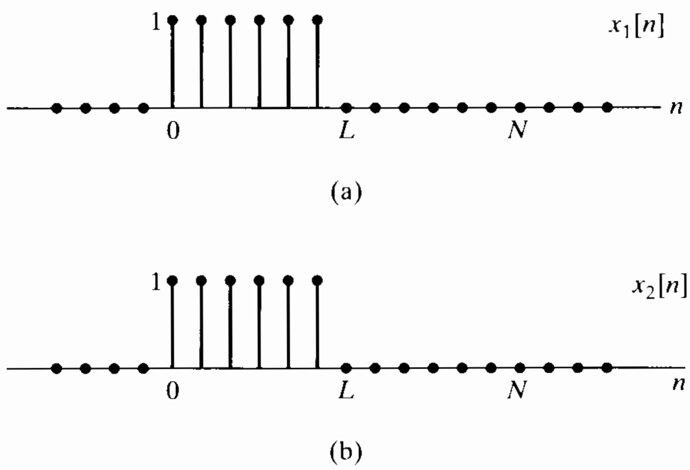

Figure 8.16 $2L$-point circular convolution of two constant sequences of length L.

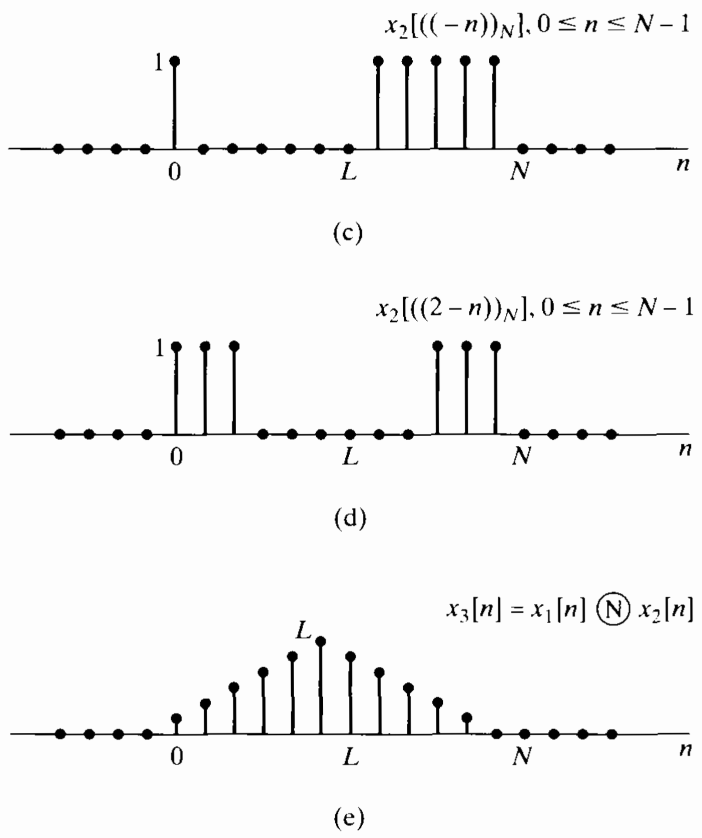

Figure 8.16 (Continued)

The circular convolution property is represented as

$$x_1[n] \circledN x_2[n] \overset{\mathcal{DFT}}{\longleftrightarrow} X_1[k]X_2[k]. \tag{8.126}$$

In view of the duality of the DFT relations, it is not surprising that the DFT of a product of two N-point sequences is the circular convolution of their respective discrete Fourier transforms. Specifically, if $x_3[n] = x_1[n]x_2[n]$, then

$$X_3[k] = \frac{1}{N}\sum_{\ell=0}^{N-1} X_1[\ell]X_2[((k-\ell))_N] \tag{8.127}$$

or

$$x_1[n]x_2[n] \overset{\mathcal{DFT}}{\longleftrightarrow} \frac{1}{N}X_1[k] \circledN X_2[k]. \tag{8.128}$$

8.6.6 Summary of Properties of the Discrete Fourier Transform

The properties of the discrete Fourier transform that we discussed in Section 8.6 are summarized in Table 8.2. Note that for all of the properties, the expressions given specify $x[n]$ for $0 \leq n \leq N-1$ and $X[k]$ for $0 \leq k \leq N-1$. Both $x[n]$ and $X[k]$ are equal to zero outside those ranges.

TABLE 8.2

Finite-Length Sequence (Length N)	N-point DFT (Length N)
1. $x[n]$	$X[k]$
2. $x_1[n], x_2[n]$	$X_1[k], X_2[k]$
3. $ax_1[n] + bx_2[n]$	$aX_1[k] + bX_2[k]$
4. $X[n]$	$Nx[((-k))_N]$
5. $x[((n-m))_N]$	$W_N^{km}X[k]$
6. $W_N^{-\ell n}x[n]$	$X[((k-\ell))_N]$
7. $\sum_{m=0}^{N-1} x_1(m)x_2[((n-m))_N]$	$X_1[k]X_2[k]$
8. $x_1[n]x_2[n]$	$\frac{1}{N}\sum_{\ell=0}^{N-1} X_1(\ell)X_2[((k-\ell))_N]$
9. $x^*[n]$	$X^*[((-k))_N]$
10. $x^*[((-n))_N]$	$X^*[k]$
11. $\mathcal{R}e\{x[n]\}$	$X_{\text{ep}}[k] = \frac{1}{2}\{X[((k))_N] + X^*[((-k))_N]\}$
12. $j\mathcal{J}m\{x[n]\}$	$X_{\text{op}}[k] = \frac{1}{2}\{X[((k))_N] - X^*[((-k))_N]\}$
13. $x_{\text{ep}}[n] = \frac{1}{2}\{x[n] + x^*[((-n))_N]\}$	$\mathcal{R}e\{X[k]\}$
14. $x_{\text{op}}[n] = \frac{1}{2}\{x[n] - x^*[((-n))_N]\}$	$j\mathcal{J}m\{X[k]\}$
Properties 15–17 apply only when $x[n]$ is real.	
15. Symmetry properties	$\begin{cases} X[k] = X^*[((-k))_N] \\ \mathcal{R}e\{X[k]\} = \mathcal{R}e\{X[((-k))_N]\} \\ \mathcal{J}m\{X[k]\} = -\mathcal{J}m\{X[((-k))_N]\} \\ \lvert X[k]\rvert = \lvert X[((-k))_N]\rvert \\ \sphericalangle\{X[k]\} = -\sphericalangle\{X[((-k))_N]\} \end{cases}$
16. $x_{\text{ep}}[n] = \frac{1}{2}\{x[n] + x[((-n))_N]\}$	$\mathcal{R}e\{X[k]\}$
17. $x_{\text{op}}[n] = \frac{1}{2}\{x[n] - x[((-n))_N]\}$	$j\mathcal{J}m\{X[k]\}$

8.7 LINEAR CONVOLUTION USING THE DISCRETE FOURIER TRANSFORM

We will show in Chapter 9 that efficient algorithms are available for computing the discrete Fourier transform of a finite-duration sequence. These are known collectively as *fast Fourier transform* (FFT) algorithms. Because these algorithms are available, it is computationally efficient to implement a convolution of two sequences by the following procedure:

(a) Compute the N-point discrete Fourier transforms $X_1[k]$ and $X_2[k]$ of the two sequences $x_1[n]$ and $x_2[n]$, respectively.

(b) Compute the product $X_3[k] = X_1[k]X_2[k]$ for $0 \le k \le N-1$.

(c) Compute the sequence $x_3[n] = x_1[n] \circledN x_2[n]$ as the inverse DFT of $X_3[k]$.

In most applications, we are interested in implementing a linear convolution of two sequences; i.e., we wish to implement a linear time-invariant system. This is certainly true, for example, in filtering a sequence such as a speech waveform or a radar signal or in computing the autocorrelation function of such signals. As we saw in Section 8.6.5, the multiplication of discrete Fourier transforms corresponds to a circular convolution of the sequences. To obtain a linear convolution, we must ensure that circular convolution has the effect of linear convolution. The discussion at the end of Example 8.11 hints at how this might be done. We now present a more detailed analysis.

8.7.1 Linear Convolution of Two Finite-Length Sequences

Consider a sequence $x_1[n]$ whose length is L points and a sequence $x_2[n]$ whose length is P points, and suppose that we wish to combine these two sequences by linear convolution to obtain a third sequence

$$x_3[n] = \sum_{m=-\infty}^{\infty} x_1[m]x_2[n-m]. \tag{8.129}$$

Figure 8.17(a) shows a typical sequence $x_1[m]$ and Figure 8.17(b) shows a typical sequence $x_2[n-m]$ for several values of n. Clearly, the product $x_1[m]x_2[n-m]$ is zero for all m whenever $n < 0$ and $n > L+P-2$; i.e., $x_3[n] \neq 0$ for $0 \le n \le L+P-2$. Therefore, $(L+P-1)$ is the maximum length of the sequence $x_3[n]$ resulting from the linear convolution of a sequence of length L with a sequence of length P.

8.7.2 Circular Convolution as Linear Convolution with Aliasing

As Examples 8.10 and 8.11 show, whether a circular convolution corresponding to the product of two N-point DFTs is the same as the linear convolution of the corresponding finite-length sequences depends on the length of the DFT in relation to the length of the finite-length sequences. An extremely useful interpretation of the relationship between circular convolution and linear convolution is in terms of time aliasing. Since this interpretation is so important and useful in understanding circular convolution, we will develop it in several ways.

In Section 8.4 we observed that if the Fourier transform $X(e^{j\omega})$ of a sequence $x[n]$ is sampled at frequencies $\omega_k = 2\pi k/N$, then the resulting sequence corresponds to the DFS coefficients of the periodic sequence

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n-rN]. \tag{8.130}$$

From our discussion of the DFT, it follows that the finite-length sequence

$$X[k] = \begin{cases} X(e^{j(2\pi k/N)}), & 0 \le k \le N-1, \\ 0, & \text{otherwise}, \end{cases} \tag{8.131}$$

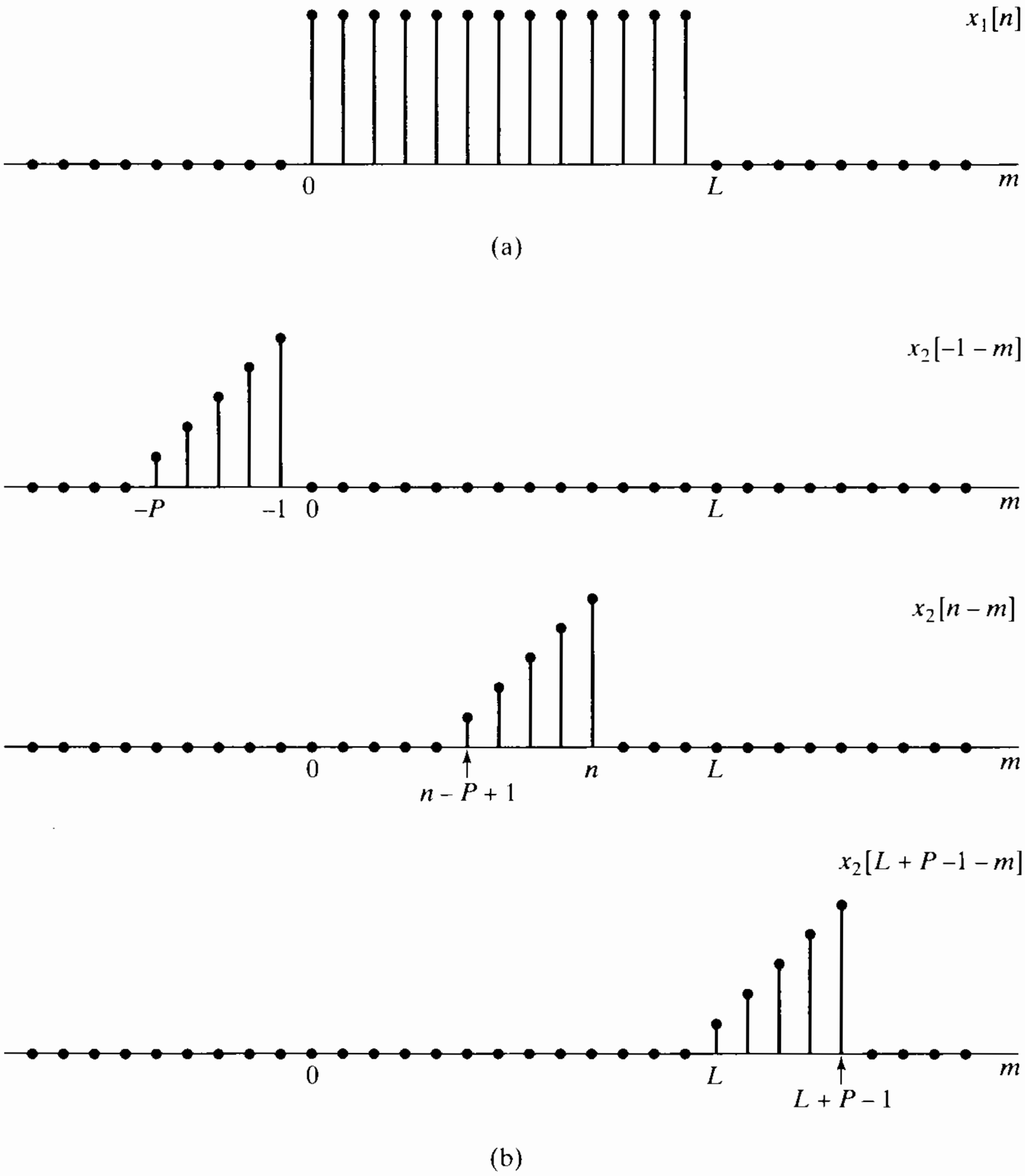

Figure 8.17 Example of linear convolution of two finite-length sequences showing that the result is such that $x_3[n] = 0$ for $n \leq -1$ and for $n \geq L + P - 1$. (a) Finite-length sequence $x_1[n]$. (b) $x_2[n - m]$ for several values of n.

is the DFT of one period of $\tilde{x}[n]$, as given by Eq. (8.130); i.e.,

$$x_p[n] = \begin{cases} \tilde{x}[n], & 0 \leq n \leq N - 1, \\ 0, & \text{otherwise}. \end{cases} \tag{8.132}$$

Obviously, if $x[n]$ has length less than or equal to N, no time aliasing occurs and $x_p[n] = x[n]$. However, if the length of $x[n]$ is greater than N, $x_p[n]$ may not be equal to $x[n]$ for some or all values of n. We will henceforth use the subscript p to denote that a sequence is one period of a periodic sequence resulting from an inverse DFT of a sampled Fourier transform. The subscript can be dropped if it is clear that time aliasing is avoided.

The sequence $x_3[n]$ in Eq. (8.129) has Fourier transform

$$X_3(e^{j\omega}) = X_1(e^{j\omega})X_2(e^{j\omega}). \tag{8.133}$$

If we define a DFT

$$X_3[k] = X_3(e^{j(2\pi k/N)}), \qquad 0 \le k \le N-1, \tag{8.134}$$

then it is clear from Eqs. (8.133) and (8.134) that, also

$$X_3[k] = X_1(e^{j(2\pi k/N)})X_2(e^{j(2\pi k/N)}), \qquad 0 \le k \le N-1, \tag{8.135}$$

and therefore,

$$X_3[k] = X_1[k]X_2[k]. \tag{8.136}$$

That is, the sequence resulting as the inverse DFT of $X_3[k]$ is

$$x_{3p}[n] = \begin{cases} \sum_{r=-\infty}^{\infty} x_3[n-rN], & 0 \le n \le N-1, \\ 0, & \text{otherwise}, \end{cases} \tag{8.137}$$

and from Eq. (8.136), it follows that

$$x_{3p}[n] = x_1[n] \circledN x_2[n]. \tag{8.138}$$

Thus, the circular convolution of two finite-length sequences is equivalent to linear convolution of the two sequences, followed by time aliasing according to Eq. (8.137).

Note that if N is greater than or equal to either L or P, $X_1[k]$ and $X_2[k]$ represent $x_1[n]$ and $x_2[n]$ exactly, but $x_{3p}[n] = x_3[n]$ for all n only if N is greater than or equal to the length of the sequence $x_3[n]$. As we showed in Section 8.7.1, if $x_1[n]$ has length L and $x_2[n]$ has length P, then $x_3[n]$ has maximum length $(L+P-1)$. Therefore, the circular convolution corresponding to $X_1[k]X_2[k]$ is identical to the linear convolution corresponding to $X_1(e^{j\omega})X_2(e^{j\omega})$ if N, the length of the DFTs, satisfies $N \ge L+P-1$.

Example 8.12 Circular Convolution as Linear Convolution with Aliasing

The results of Example 8.15 are easily understood in light of the interpretation just discussed. Note that $x_1[n]$ and $x_2[n]$ are identical constant sequences of length $L = P = 6$, as shown in Figure 8.18(a). The linear convolution of $x_1[n]$ and $x_2[n]$ is of length $L+P-1 = 11$ and has the triangular shape shown in Figure 8.18(b). In Figures 8.18(c) and (d) are shown two of the shifted versions $x_3[n-rN]$ in Eq. (8.137), $x_3[n-N]$ and $x_3[n+N]$ for $N = 6$. The N-point circular convolution

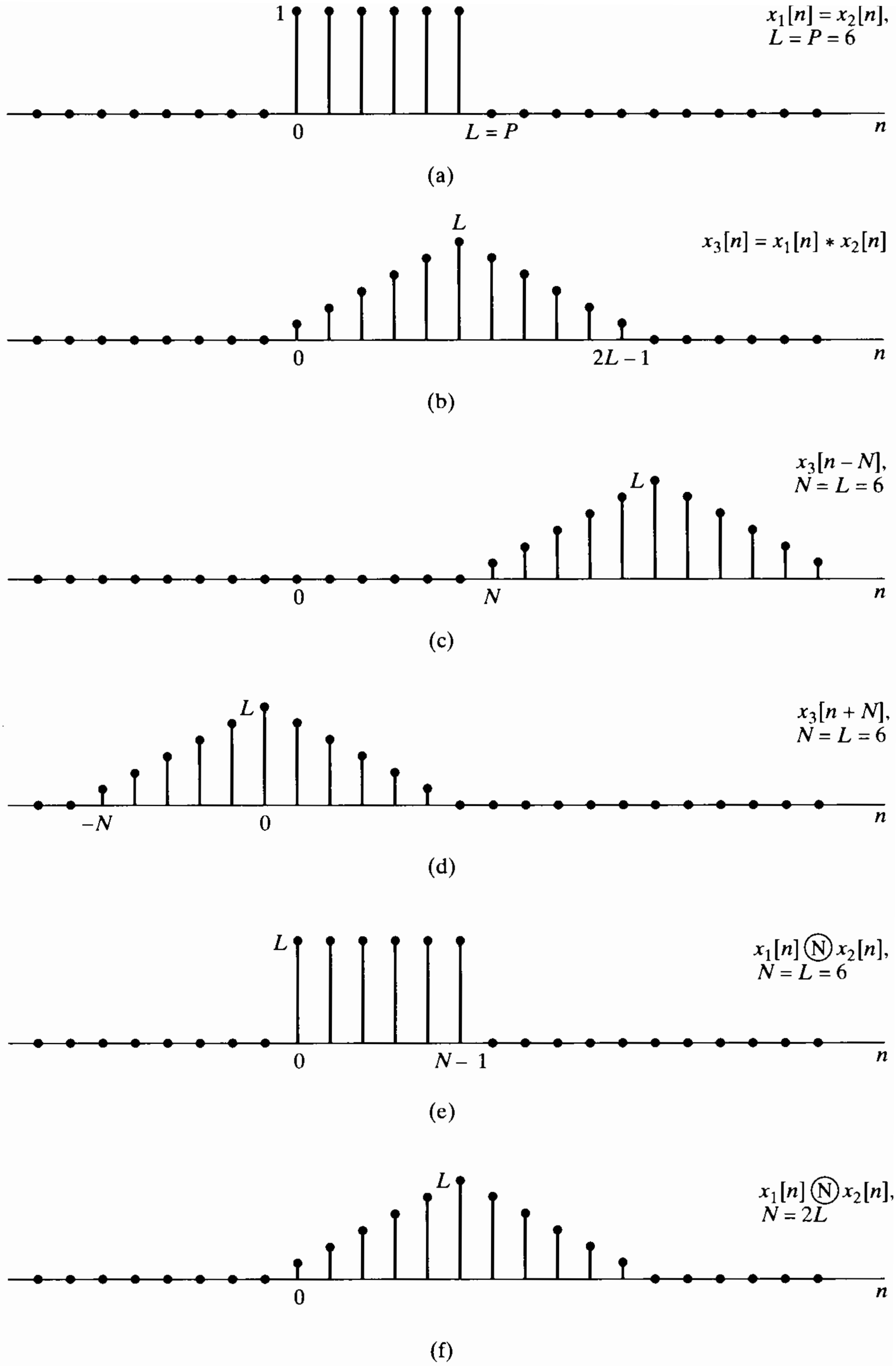

Figure 8.18 Illustration that circular convolution is equivalent to linear convolution followed by aliasing. (a) The sequences $x_1[n]$ and $x_2[n]$ to be convolved. (b) The linear convolution of $x_1[n]$ and $x_2[n]$. (c) $x_3[n - N]$ for $N = 6$. (d) $x_3[n + N]$ for $N = 6$. (e) $x_1[n]$ ⑥ $x_2[n]$, which is equal to the sum of (b), (c), and (d) in the interval $0 \le n \le 5$. (f) $x_1[n]$ ⑫ $x_2[n]$.

of $x_1[n]$ and $x_2[n]$ can be formed by using Eq. (8.137). This is shown in Figure 8.18(e) for $N = L = 6$ and in Figure 8.18(f) for $N = 2L = 12$. Note that for $N = L = 6$, only $x_3[n]$ and $x_3[n + N]$ contribute to the result. For $N = 2L = 12$, only $x_3[n]$ contributes to the result. Since the length of the linear convolution is $(2L - 1)$, the result of the circular convolution for $N = 2L$ is identical to the result of linear convolution for all $0 \leq n \leq N - 1$. In fact, this would be true for $N = 2L - 1 = 11$ as well.

As Example 8.12 points out, time aliasing in the circular convolution of two finite-length sequences can be avoided if $N \geq L + P - 1$. Also, it is clear that if $N = L = P$, all of the sequence values of the circular convolution may be different from those of the linear convolution. However, if $P < L$, some of the sequence values in an L-point circular convolution will be equal to the corresponding sequence values of the linear convolution. The time-aliasing interpretation is useful for showing this.

Consider two finite-duration sequences $x_1[n]$ and $x_2[n]$, with $x_1[n]$ of length L and $x_2[n]$ of length P, where $P < L$, as indicated in Figures 8.19(a) and 8.19(b), respectively. Let us first consider the L-point circular convolution of $x_1[n]$ and $x_2[n]$ and inquire as to which sequence values in the circular convolution are identical to values that would be obtained from a linear convolution and which are not. The linear convolution of $x_1[n]$ with $x_2[n]$ will be a finite-length sequence of length $(L + P - 1)$, as indicated in Figure 8.19(c). To determine the L-point circular convolution, we use Eqs. (8.137) and

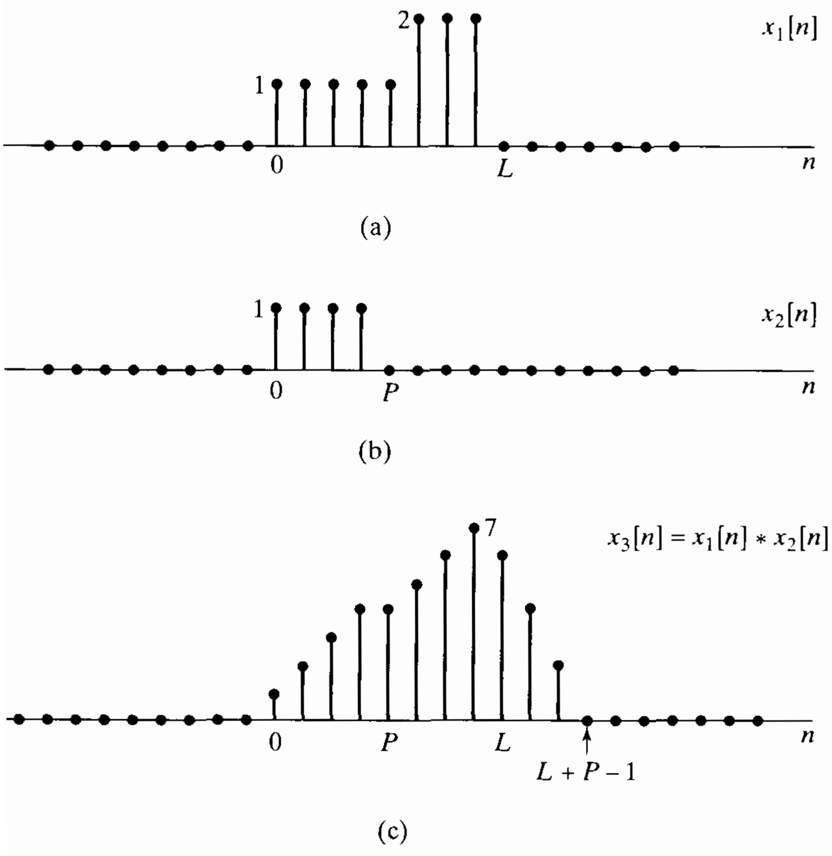

Figure 8.19 An example of linear convolution of two finite-length sequences.

(8.138) so that

$$x_{3p}[n] = \begin{cases} x_1[n] \circledL x_2[n] = \sum_{r=-\infty}^{\infty} x_3[n - rL], & 0 \le n \le L-1, \\ 0, & \text{otherwise.} \end{cases} \quad (8.139)$$

Figure 8.20(a) shows the term in Eq. (8.139) for $r = 0$, and Figures 8.20(b) and 8.20(c) show the terms for $r = -1$ and $r = +1$, respectively. From Figure 8.20, it should be clear that in the interval $0 \le n \le L-1$, $x_{3p}[n]$ is influenced only by $x_3[n]$ and $x_3[n+L]$.

In general, whenever $P < L$, only the term $x_3[n + L]$ will alias into the interval $0 \le n \le L-1$. More specifically, when these terms are summed, the last $(P-1)$ points of $x_3[n+L]$, which extend from $n = 0$ to $n = P-2$, will be added to the first $(P-1)$ points of $x_3[n]$, and the last $(P-1)$ points of $x_3[n]$, extending from $n = L$ to $n = L + P - 2$, will be discarded. Then $x_{3p}[n]$ is formed by extracting the portion for $0 \le n \le L-1$. Since the last $(P-1)$ points of $x_3[n+L]$ and the last $(P-1)$ points of $x_3[n]$ are identical, we can alternatively view the process of forming the circular convolution $x_{3p}[n]$ through linear convolution plus aliasing as taking the $(P-1)$ values of $x_3[n]$ from $n = L$ to $n = L + P - 2$ and adding them to the first $(P-1)$ values of $x_3[n]$. This process is illustrated in Figure 8.21 for the case $P = 4$ and $L = 8$. Figure 8.21(a) shows the linear convolution $x_3[n]$, with the points for $n \ge L$ denoted by open symbols. Note that only $(P-1)$ points for $n \ge L$ are nonzero. Figure 8.21(b) shows the formation of $x_{3p}[n]$ by "wrapping $x_3[n]$ around on itself." The first $(P-1)$ points are corrupted by the time aliasing, and the remaining points from $n = P-1$ to $n = L-1$ (i.e., the last $L - P + 1$ points) are *not* corrupted; that is, they are identical to what would be obtained with a linear convolution.

From this discussion, it should be clear that if the circular convolution is of sufficient length relative to the lengths of the sequences $x_1[n]$ and $x_2[n]$, then aliasing with nonzero values can be avoided, in which case the circular convolution and linear convolution will be identical. Specifically, if, for the case just considered, $x_3[n]$ is replicated with period $N \ge L + P - 1$, then no nonzero overlap will occur. Figures 8.21(c) and 8.21(d) illustrate this case, again for $P = 4$ and $L = 8$, with $N = 11$.

8.7.3 Implementing Linear Time-Invariant Systems Using the DFT

The previous discussion focused on ways of obtaining a linear convolution from a circular convolution. Since linear time-invariant systems can be implemented by convolution, this implies that circular convolution (implemented by the procedure suggested at the beginning of Section 8.7) can be used to implement these systems. To see how this can be done, let us first consider an L-point input sequence $x[n]$ and a P-point impulse response $h[n]$. The linear convolution of these two sequences, which will be denoted by $y[n]$, has finite duration with length $(L + P - 1)$. Consequently, as discussed in Section 8.7.2, for the circular convolution and linear convolution to be identical, the circular convolution must have a length of at least $(L + P - 1)$ points. The circular convolution can be achieved by