SIMULATION SYSTEMS

TRANSFORMATION METHOD

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Transformation Method

- Fundamental transformation law of probabilities
  - If we sample a random variable from some density f(x), then apply a function y(x) to x, the density g(y) of y will be related to that of x by the following rule:

$$g(y) = f(x)\left|\frac{dx}{dy}\right|.$$

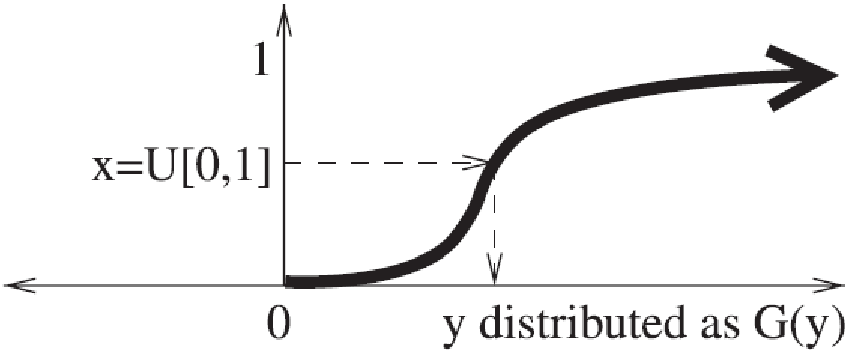
- If we know how to sample from some continuous density function f (x), we can use this transformation to sample from some other continuous density function g(y)

Transformation Method

- Let x~U[0,1]

$$\frac{dx}{dy} = g(y) \Rightarrow x = \int_{-\infty}^{y} g(u)\,du = G(y) \Rightarrow y = G^{-1}(x)$$

- We find the distribution G(y) by integrating the desired density g(y), invert the distribution to get G^{-1}, and apply this inverse distribution to x to get y distributed according to g(y).



Transformation Method: Example

- Sampling from an exponential distribution with parameter λ:

$$g(y) = \begin{cases} 0 & y < 0 \\ \lambda e^{-\lambda y} & y \geq 0 \end{cases}$$

$$G(y) = \int_{-\infty}^{y} g(u)\, du = \int_{0}^{y} g(u)\, du = -e^{-\lambda u}\Big|_0^y = 1 - e^{-\lambda y}.$$

$$x = 1 - e^{-\lambda y} \quad \Rightarrow \quad \boxed{y = -\frac{1}{\lambda} \ln|1 - x|.}$$

  - Generate x~U[0,1] then use the above transformation

Transformation Method for Joint Distributions

- Modified version of the fundamental transformation law of probabilities:

$$g(y_1, \ldots, y_k) = f(x_1, \ldots, x_k) \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \cdots & \frac{\partial x_1}{\partial y_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_k}{\partial y_1} & \cdots & \frac{\partial x_k}{\partial y_k} \end{vmatrix}$$

Transformation Method for Joint Distributions

- Assuming each x_i is an independent U[0,1], we can sample for y_is if we can find a set of functions:

$$
\begin{array}{ccc}
y_1(x_1, \ldots, x_k) & & x_1(y_1, \ldots, y_k) \\
y_2(x_1, \ldots, x_k) & \xrightarrow{\text{Invert}} & x_2(y_1, \ldots, y_k) \\
\vdots & & \vdots \\
y_2(x_1, \ldots, x_k) & & x_2(y_1, \ldots, y_k)
\end{array}
$$

$$
\Rightarrow \quad \left| \begin{array}{ccc} \frac{\partial x_1}{\partial y_1} & \cdots & \frac{\partial x_1}{\partial y_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_k}{\partial y_1} & \cdots & \frac{\partial x_k}{\partial y_k} \end{array} \right| = g(y_1, \ldots, y_k)
$$

Transformation Method for Joint Distributions

- This knowledge is not very helpful in figuring out what the transformation functions practically
  - Difficult to integrate many practical probability density functions
  - Difficult to invert multidimensional functions
- Still useful to give us a way to prove whether a transformation will or will not work
  - That is, take a guess and verify that it satisfies the

Box–Muller Method

- Technique for sampling normal variables
- Desired: $$g(y_1, y_2) = \frac{1}{\sqrt{2\pi}} e^{-y_1^2/2} \frac{1}{\sqrt{2\pi}} e^{-y_2^2/2}$$
- Separable: $g(y_1, y_2) = g_1(y_1) g_2(y_2)$
- Let $x_1 \sim U[0,1]$ and $x_2 \sim U[0,1]$, then the following transformation can be used to do the job:

$$y_1(x_1, x_2) = \sqrt{-2 \ln x_1} \cos(2\pi x_2)$$

$$y_2(x_1, x_2) = \sqrt{-2 \ln x_1} \sin(2\pi x_2).$$

Box–Muller Method: Verification

$$y_1^2 + y_2^2 = -2\ln x_1(\cos^2(2\pi x_2) + \sin^2(2\pi x_2)) = -2\ln x_1 \Rightarrow x_1 = e^{-(y_1^2+y_2^2)/2}$$

$$\frac{y_2}{y_1} = \tan(2\pi x_2) \Rightarrow x_2 = \frac{1}{2\pi}\arctan\frac{y_2}{y_1}.$$

Therefore

$$\begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix} = \begin{vmatrix} -2y_1 e^{-(y_1^2+y_2^2)/2} & -2y_1 e^{-(y_1^2+y_2^2)/2} \\ -\frac{y_2}{y_1^2}\frac{1}{2\pi}\frac{y_1^2}{y_1^2+y_2^2} & \frac{1}{y_1}\frac{1}{2\pi}\frac{y_1^2}{y_1^2+y_2^2} \end{vmatrix}$$

$$= -\frac{1}{2\pi}e^{-(y_1^2+y_2^2)/2}\left[1 + \frac{y_2^2}{y_1^2}\right]\left[\frac{y_1^2}{y_1^2+y_2^2}\right] = -\frac{1}{2\pi}e^{-(y_1^2+y_2^2)/2}$$

$$= \frac{1}{\sqrt{2\pi}}e^{-y_1^2/2}\frac{1}{\sqrt{2\pi}}e^{-y_2^2/2}.$$

Assignments

- Generate samples from a random variable of exponential distribution with a parameter $\lambda=2$ and verify the output using histograms.
- Generate samples from a 2D normal random variable of unity standard deviations using Box-Muller method.