Medical Image Reconstruction
Term II – 2012

Topic 6: Tomography

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Tomography

- The Greek word “tomos” means a section, a slice, or a cut.
- Tomography is the process of imaging a cross section.
- Particularly useful in medical imaging.
  - Nobody wants to be cut open to see what is inside!
**Example**

- Can you compute the locations of the trees from 2 images?
  - **Answer:** Yes
Example

Can you compute the values of this matrix given its projections?

Answer: Yes

\[
\begin{align*}
x_1 + x_2 &= 5, & x_1 &= 3, \\
x_3 + x_4 &= 4, & x_2 &= 2, \\
x_1 + x_3 &= 7, & x_3 &= 4, \\
x_2 + x_4 &= 2. & x_4 &= 0.
\end{align*}
\]
Projection

- Also termed *ray sum, line integral, or Radon transform*
Sinogram

- Displays angle dependence of projections
  - Example: point source on the y-axis to further illustrate the angle $\theta$ dependency of the projection $p(s, \theta)$
Projection of Discrete Object

- Projections are weighted by the line-length within each pixel

\[ p(i, \theta) = a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 + a_{i4}x_4, \quad i = 1, 2, 3, 4. \]
In image reconstruction, we not only need to find the location but also the intensity value of the object of interest.

- **Backprojection**
Backprojection Example

- Blurred version of original

Projection

Backprojection

- $b_1 = 5 + 7 = 12$
- $b_2 = 5 + 2 = 7$
- $b_3 = 4 + 7 = 11$
- $b_4 = 4 + 2 = 6$
Backprojection in Algebraic Form

- Projection $P$
  \[ P = AX, \quad X = [x_1, x_2, x_3, x_4]^T \]
  \[ A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \]
  \[ P = [p(1, 0^\circ), p(2, 0^\circ), p(1, 270^\circ), p(2, 270^\circ)]^T = [7, 2, 5, 4]^T. \]

- Backprojection: use adjoint operator $A^T$
  \[ B = A^T P = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}^T \begin{bmatrix} 7 \\ 2 \\ 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 7 \\ 2 \\ 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 12 \\ 7 \\ 11 \\ 6 \end{bmatrix}. \]
Radon Transform

- Computes parallel projections at specific angles

\[ f(x,y) \rightarrow p(s, \theta) \]

\[ p(s, \theta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \delta(x \cos \theta + y \sin \theta - s) \, dx \, dy, \]

\[ p(s, \theta) = \int_{-\infty}^{\infty} f(s \cos \theta - t \sin \theta, s \sin \theta + t \cos \theta) \, dt, \]

\[ \mathcal{R} f(t, \theta) := \int_{\ell_{t, \theta}} f \, ds = \int_{s=-\infty}^{\infty} f(t \cos(\theta) - s \sin(\theta), t \sin(\theta) + s \cos(\theta)) \, ds. \]
Inverse Radon Transform

- Parallel Beam Backprojection

\[ p(s, \theta) \rightarrow f(x, y) \]

\[
\begin{align*}
  b(x, y) &= \int_0^{\pi} p(s, \theta) \bigg|_{s=x \cos \theta + y \sin \theta} \, d\theta, \\
  b(x, y) &= \int_0^{\pi} p(x \cdot \theta, \theta) \, d\theta,
\end{align*}
\]
Backprojection in Frequency Domain

- Each view adds a line in the Fourier space
  - Central area in k-space has higher sample density and results in effective lowpass filtered spectrum
Filtered Backprojection

- To counter this blurring effect, we must compensate for the non-uniformity in the Fourier space.

- **Density in the Fourier space is proportional to:**

  \[
  \frac{1}{\sqrt{\omega_x^2 + \omega_y^2}}.
  \]

- **Solution:** multiply k-space by ramp filter,
  - Method 1: Filter individual projections
  - Method 2: Filter whole Image

- **Practical Realization**

  - Ram-Lak Filter
  - Shepp-Logan Filter
Define Hilbert Transform as:
\[
\mathcal{H} g(t) = \mathcal{F}^{-1} \left[ i \cdot \text{sgn}(\omega) \cdot \mathcal{F} g(\omega) \right] (t).
\]

\[
f(x, y) = -\frac{1}{2} B \left\{ \mathcal{F}^{-1} \left[ i \cdot \text{sgn}(S) \cdot \mathcal{F} \left( \frac{\partial (\mathcal{R} f)(t, \theta)}{\partial t} \right) (S, \theta) \right] \right\} (x, y).
\]
## Parallel Beam Reconstruction Methods

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Fan Beam Reconstruction Problem

- Almost all present CT systems use fan beam rather than parallel beam projections
  - Much more efficient: faster acquisition, lower patient dose
Fan Beam Reconstruction Problem

- rebin every fan-beam ray into a parallel-beam ray. For each fan-beam ray-sum \( g(\gamma, \beta) \), we can find a parallel beam ray-sum \( p(s, \theta) \) that has same orientation as the fan-beam ray with the relations,

\[
\theta = \gamma + \beta, \quad s = D \sin \gamma, \quad \Rightarrow \quad p(s, \theta) = g(\gamma, \beta).
\]
Fan Beam Reconstruction Problem

A Parallel-Beam Algorithm

\[ f(x, y) = \int \int \{ p(s, \theta) \text{ and other things with } s \text{ and } \theta \} ds d\theta \]

\[ p(s, \theta) = g(\gamma, \beta) \]

\[ \theta = \gamma + \beta \text{ and } s = D \sin \gamma \]

Jacobian \( J(\gamma, \beta) \)

Substituting Parallel-Beam Data with Fan-Beam Data

Changing Variables

\[ f(x, y) = \int \int \{ g(\gamma, \beta) \text{ and other things with } \gamma \text{ and } \beta \} J(\gamma, \beta) d\gamma d\beta \]

A Fan-Beam Algorithm
Algebraic Reconstruction Technique

\[ AX = P, \quad X = A^+ P = V \Sigma^+ U^T P. \]

\[ x_{next} = x_{current} - \text{Backproject}_{ray} \left\{ \frac{\text{Project}_{ray}(x_{current}) - \text{Measurement}_{ray}}{\text{Normalization Factor}} \right\}. \]
Other Algebraic Methods

- Gradient descent

\[ \mathbf{x}_{\text{next}} = \mathbf{x}_{\text{current}} - \alpha_{\text{current}} \Delta(\mathbf{x}_{\text{current}}), \]

- Maximum-Likelihood Expectation-Maximization

\[ \mathbf{x}_{\text{next}} = \mathbf{x}_{\text{current}} \frac{\text{Backproject } \left\{ \frac{\text{Measurement}}{\text{Project } (\mathbf{x}_{\text{current}})} \right\}}{\text{Backproject } \{1\}}, \]
Advanced Tomography Problems

- Reconstruction from incomplete data
  - Truncated projections
  - Limited-angle projections
  - Exterior data
  - ROI reconstruction
- Extension to 3D
- Stability of reconstruction
Matlab Functions

- Look up the help for the following function:
  - radon
  - iradon
  - fan2para
  - fanbeam
  - ifanbeam
  - para2fan
  - phantom
Exercise

P1. For a medical image of your choice:
   - A. Generate the Radon transform and display its sinogram
   - B. Reconstruct the image back from its projections
   - C. Compare the two images and record the error for different numbers of projections and provide your comments.

P2. Repeat the above problem P1 for a fan beam system (rather than the parallel beam system in P1). Also, compare the parallel beams obtained from rebinning the fan rays and comment on what you found.

P3. Compare different filtering strategies and provide your choice of the best methodology based on an experimental study. (study filter type and support, 1D vs. 2D implementation, etc.)

P4. Compare the parallel beam projections obtained from a Shepp-Logan phantom to the ones you generate using the analytical Shepp-Logan phantom and comment on the results.

P5. Do a literature review on ONE advanced tomography problem and come up with a 1 page summary of the state of the art and a comprehensive list of references.